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Standard Subgroups Isomorphic to $PSU(5, 2)$

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1. INTRODUCTION

A subgroup K of a finite group G is said to be *tightly embedded* in G if $|K|$ is even but $|K \cap K^g|$ is odd for $g \in G - N_G(K)$. A *standard* subgroup of G is a quasisimple subgroup L of G such that $K = C_G(L)$ is tightly embedded in G, $N_G(L) = N_G(K)$, and $[L, L^g] \neq 1$ for $g \in G$.

In [14], Miyamoto has classified all finite groups which contain a standard subgroup L with $L/Z(L) \cong PSU(5, 2^n)$, $n > 1$, and such that $C_G(L)$ has a cyclic Sylow 2-subgroup. In this paper we treat the case $n = 1$. More precisely, the following theorem is proved.

THEOREM. *Let G be a finite group which possesses a standard subgroup L isomorphic to $PSU(5, 2)$. Assume that $C_G(L)$ has a cyclic Sylow 2-subgroup and that $LO(G) \ntrianglelefteq G$. Then either*

(1) $E(G) \cong PSL(5, 4)$, *or*

(2) $E(G) \cong PSU(5, 2) \times PSU(5, 2)$.

The method used in proving the theorem is similar to that used by Gomi in [8] and by the author in [20]. Let t be an involution of $C_G(L)$ and set $H = C_G(t)$. Then $L \lhd H$ by our hypotheses. In Section 4, the proof of the theorem begins with a study of the fusion of the involution t. An important point is the use of the structure of the centralizer of a certain element of order 3. The group $L \cong PSU(5, 2)$ has an element w of order 3 such that $C_L(w) \cong Z_3 \times PSU(4, 2)$. Applying a theorem of Gomi [8], we can show that $K_0 = E(C_G(w))$ is isomorphic to either $PSL(4, 4)$ or $PSU(4, 2) \times PSU(4, 2)$. If $K_0 \cong PSL(4, 4)$, we distinguish two cases: $H \neq LC_H(L)$ and $H = LC_H(L)$. In Section 5, we treat the case $H \neq LC_H(L)$. The method used in this section was suggested by K. Gomi. In Section 6, we handle the case $H = LC_H(L)$ in a similar way as in [7, Section 8]. Finally, in Section 7, we treat the case $K_0 \cong PSU(4, 2) \times PSU(4, 2)$.

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Our notation is fairly standard and tends to follow the notation of [9] and [20]. In particular, for a group X, $m(X)$ denotes the 2-rank of X and $I(X)$ denotes the set of involutions of X. Moreover, for a 2-group Q, $\mathscr{E}^*(Q)$ is the set of maximal elementary abelian subgroups of Q, $\mathscr{A}(Q)$ is the set of abelian subgroups of Q of maximal order, $J_0(Q) = \langle A \mid A \in \mathscr{A}(Q)\rangle$, $J_r(Q)$ is the subgroup generated by all abelian subgroups of Q of maximal rank, and $J_e(Q)$ is the subgroup generated by all elementary abelian subgroups of Q of maximal order.

2. Preliminaries

In this section we collect some preliminary lemmas.

(2.1) *Let X be a subset of a group G and P a Sylow p-subgroup of $C_G(X)$. Assume that a subgroup B of P is weakly closed in P with respect to G. If a subset X_1 of $C_G(B)$ is conjugate to X in G, then X and X_1 are conjugate in $N_G(B)$.*

Proof. Choose an element $g \in G$ so that $X_1 = X^g$. Then $B \leqslant C_G(X)^g$ and $B^h \leqslant P^g$ for some $h \in C_G(X)^g$. The weak closure of B implies $B^{hg^{-1}} = B$, so $gh^{-1} \in N_G(B)$ and $X^{gh^{-1}} = X_1$.

The next lemma is well known.

(2.2) *Let P be a Sylow p-subgroup of a group G and x an element of P. Assume that $|C_P(y)| \leqslant |C_P(x)|$ for all $y \in x^G \cap P$.* (In that case x is said to be *extremal* in P with respect to G.) *Then the following conditions hold.*

(1) $C_P(x) \in \mathrm{Syl}_p(C_G(x))$.

(2) *If $y \in x^G \cap P$, then there exists an element g in G such that $y^g = x$ and $C_P(y)^g \leqslant C_P(x)$.*

Proof. Let Q be a Sylow p-subgroup of $C_G(x)$ containing $C_P(x)$ and h an element of G such that $Q \leqslant P^h$. Then $C_P(x) \leqslant Q = C_{P^h}(x)$ and $|C_P(x)| \leqslant |C_P(x^{h^{-1}})|$. Since $x^{h^{-1}} \in x^G \cap P$, $C_P(x) = Q$ and (1) holds. Suppose $y \in x^G \cap P$ and $y^a = x$, $a \in G$. There exists $b \in C_G(x)$ such that $C_{P^a}(x) \leqslant C_P(x)^b$ by (1). Setting $g = ab^{-1}$, we have $y^g = x$ and $C_P(y)^g = C_{P^a}(x)^{b^{-1}} \leqslant C_P(x)$.

(2.3) *Suppose that a group $X \cong SL(2, 4)$ acts nontrivially on an elementary abelian group E of order 16. If P is a Sylow 2-subgroup of X, then either*

(1) *$|C_E(P)| = 4$ and E is a natural module for $X \cong SL(2, 4)$, or*

(2) *$|C_E(P)| = 2$ and E is a natural module for $X \cong A_5$.*

In Case (1), X acts transitively on $E^{\#}$ but in Case (2), $E^{\#}$ decomposes under the action of X into two orbits of length 5 and 10.

Proof. See [8, Lemma (1J)].

(2.4) (1) *If* $K \cong PSL(3, 4)$, *then each involution in* $\mathrm{Aut}(K) - K$ *is conjugate to* f, g, *or* fg, *where* f *and* g *denote respectively an involutive field automorphism and a graph automorphism with* $[f, g] = 1$. *Moreover*, $C_K(f) \cong PSL(3, 2)$, $C_K(g) \cong SL(2, 4)$, *and* $C_K(fg) \cong PSU(3, 2)$.

(2) *If* $K \cong PSL(4, 4)$, *then* $\mathrm{Aut}(K) = K\langle f, g\rangle$, *where* f *and* g *denote respectively an involutive field automorphism and a graph automorphism with* $[f, g] = 1$. *All involutions in* Kf *are conjugate to* f *and all involutions in* Kfg *are conjugate to* fg. *Let* d *be a nonidentity element in the center of a Sylow* 2-*subgroup of* K *such that* $[d, g] = 1$. *Then each involution in* Kg *is conjugate to either* g *or* gd. *Moreover*, $C_K(f) \cong PSL(4, 2)$, $C_K(fg) \cong PSU(4, 2)$, $C_K(g) \cong Sp(4, 4)$, *and* $C_K(gd) = C_K(g) \cap C_K(d)$, *which is isomorphic to the centralizer of a transvection in* $Sp(4, 4)$.

Proof. See Section 19 of [2].

We will enumerate some properties of $K = PSL(4, 4)$. A detailed description of the group $PSL(4, 2^n)$ can be found in Suzuki [17] and we follow it. The set P of matrices

$$x = \begin{pmatrix} 1 & & & \\ \alpha_1 & 1 & & \\ \alpha_4 & \alpha_2 & 1 & \\ \alpha_6 & \alpha_5 & \alpha_3 & 1 \end{pmatrix}$$

forms a Sylow 2-subgroup of K. Define

M: the set of matrices x with $\alpha_2 = 0$;
Y: the set of matrices x with $\alpha_1 = \alpha_3 = 0$;
B_1: the set of matrices x with $\alpha_2 = \alpha_3 = \alpha_5 = 0$;
B_2: the set of matrices x with $\alpha_1 = \alpha_2 = \alpha_4 = 0$;
U_λ: the set of matrices x with $\alpha_2 = \alpha_3 + \alpha_1\lambda = \alpha_5 + \alpha_4\lambda = 0$,

where $0 \neq \lambda \in GF(4)$. In addition, we set

$$a = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix}.$$

(2.5) *The following conditions hold.*

(1) $J_r(P) = Y$ *and* $\mathscr{E}^*(P/Z(P)) = \{M/Z(P), Y/Z(P)\}$. *Moreover*, Y *is elementary abelian and* $M' = Z(M) = Z(P)$.

(2) *The maximal elementary abelian subgroups of M are B_1, B_2, U_λ; $0 \neq \lambda \in GF(4)$, and $Z_2(P)^g$; $g \in N_K(M)$.*

(3) *$N_K(M)$ acts transitively on the set $\{U_\lambda \mid 0 \neq \lambda \in GF(4)\}$.*

(4) *B_1 and B_2 are normal in $N_K(M)$ and U_λ is normal in $N_K(M)'$. Moreover, $U_\lambda/Z(P)$ is a natural module for $N_K(M)'/M \simeq SL(2, 4)$.*

Proof. We can verify (1), (3), and (4) by matrix multiplications. Each involution of $M - Z(P)$ is conjugate in $N_K(M)$ to one of the following three involutions:

$$b_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & & & 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 1 & & 1 \end{pmatrix}, \qquad \text{or} \qquad b.$$

Furthermore, $\mathscr{E}^*(C_M(b_1)) = \{B_1, Z_2(P)\}$, $\mathscr{E}^*(C_M(b_2)) = \{B_2, Z_2(P)\}$, and $\mathscr{E}^*(C_M(b)) = \{U_1, Z_2(P)\}$ by the structure of P. Hence (2) holds.

(2.6) *Each involution of K is conjugate to either a or b, and b is a noncentral involution. Let $w \in Y$ be a noncentral involution of K and set $H = C_K(w)$. Then*

(1) *Y is the unique elementary abelian subgroup of H of order 2^8,*

(2) *$H/Y \simeq SL(2, 4)$, H' is perfect, $|H : H'| = 4$, and $C_Y(H) = Z(H')$ is of order 4.*

Proof. By (2.5)(1), Y is weakly closed in P with respect to K. Hence w and b are conjugate in $N_K(Y)$ and we may assume that $w = b$. Denote by X the subgroup of K consisting of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \qquad A \in SL(2, 4).$$

Then $X \simeq SL(2, 4)$ and $H = C_K(b)$ is a split extension of Y by X. If Q is a Sylow 2-subgroup of H, then $J_r(Q) = Y$ by (2.5)(1), and (1) holds. (2) can be easily verified.

3. Properties of $PSU(5, 2)$

In this section we enumerate some properties of the group $PSU(5, 2)$ and its automorphisms. An excellent description of the five-dimensional unitary group $PSU(5, 2^n)$ can be found in Thomas [18]. Proofs are mostly omitted in the case where the assertions are consequences of straightforward calculations.

Let $L = PSU(5, 2)$. We identify L with $SU(5, 2)$. Let $\Gamma = GF(4)$ and $\Gamma_0 = GF(2)$. Denote α^2 by $\bar{\alpha}$ for $\alpha \in \Gamma$. The group L is the set of all 5×5 matrices x over Γ satisfying

$$x \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} {}^t\bar{x} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \quad \text{and} \quad \det x = 1,$$

where ${}^t\bar{x}$ denotes the matrix obtained from x by replacing each entry α_{ij} with $\bar{\alpha}_{ji}$. Let ω be a primitive cube root of unity in Γ. Define

$$x_1(\alpha, \beta) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & \alpha & 1 & & \\ & \beta & \bar{\alpha} & 1 & \\ & & & & 1 \end{pmatrix}, \qquad x_2(\alpha) = \begin{pmatrix} 1 & & & & \\ \alpha & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & \bar{\alpha} & 1 \end{pmatrix},$$

$$x_3(\alpha) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \alpha & & & 1 & \\ & \bar{\alpha} & & & 1 \end{pmatrix}, \qquad x_4(\alpha, \beta) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ \alpha & & 1 & & \\ & & & 1 & \\ \beta & & \bar{\alpha} & & 1 \end{pmatrix},$$

$$u = \begin{pmatrix} 1 & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega & & \\ & & & \omega & \\ & & & & 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} \omega & & & & \\ & \omega & & & \\ & & \bar{\omega} & & \\ & & & \omega & \\ & & & & \omega \end{pmatrix}.$$

Furthermore, we set

$$\begin{aligned} X_1 &= \{x_1(\alpha, \beta) \mid \alpha, \beta \in \Gamma, \alpha\bar{\alpha} = \beta + \bar{\beta}\}, \\ X_2 &= \{x_2(\alpha) \mid \alpha \in \Gamma\}, \\ X_3 &= \{x_3(\alpha) \mid \alpha \in \Gamma\}, \\ X_4 &= \{x_4(\alpha, \beta) \mid \alpha, \beta \in \Gamma, \alpha\bar{\alpha} = \beta + \bar{\beta}\}. \end{aligned}$$

(3.1) The subgroup $S = X_1X_2X_3X_4$ has order 2^{10} and it is a Sylow 2-subgroup of L. Each element of S has a unique expression as

$$x = x_1(\alpha_1, \beta_1)\, x_2(\alpha_2)\, x_3(\alpha_3)\, x_4(\alpha_4, \beta_4).$$

The multiplication within X_i is given by

$$\begin{aligned} x_i(\alpha, \beta)\, x_i(\gamma, \delta) &= x_i(\alpha + \gamma, \beta + \delta + \bar{\alpha}\gamma), && i = 1, 4,\\ x_i(\alpha)\, x_i(\beta) &= x_i(\alpha + \beta), && i = 2, 3. \end{aligned}$$

Hence, X_1 and X_4 are quaternion with $Z(X_i) = \{x_i(0, \beta) \mid \beta \in \Gamma_0\}$ and X_2 and X_3 are elementary abelian of order 4. The commutator relations between elements of the X_i are

$$\begin{aligned} [x_1(\alpha, \beta), x_2(\gamma)] &= x_3(\bar{\beta}\gamma)\, x_4(\alpha\gamma, \beta\gamma\bar{\gamma}),\\ [x_1(\alpha, \beta), x_4(\gamma, \delta)] &= x_3(\bar{\alpha}\gamma),\\ [x_2(\alpha), x_3(\beta)] &= x_4(0, \alpha\bar{\beta} + \bar{\alpha}\beta). \end{aligned}$$

All other types of commutators are trivial. The structure of S is completely determined by these formulas.

(3.2) $N_L(S) = \langle e_1, e_2\rangle S$ and $\langle e_1, e_2\rangle$ is elementary abelian of order 9 and normalizes each X_i.

(3.3) The subgroup $\langle u, v\rangle$ is dihedral of order 8 and normalizes $\langle e_1, e_2\rangle$. Moreover, u and v transform the elements of S according to the formulas:

$$\begin{aligned} ux_2(\alpha)u &= x_3(\alpha), \qquad & ux_4(\alpha, \beta)u &= x_4(\alpha, \beta),\\ vx_1(\alpha, \beta)v &= x_4(\alpha, \beta), \qquad & vx_3(\alpha)v &= x_3(\bar{\alpha}), \end{aligned}$$

$$ux_1(\alpha, \beta)u = x_1(\alpha\beta^{-1}, \beta^{-1})\, h_1ux_1(\alpha\beta, \beta^{-1}),$$

$$vx_2(\alpha)v = x_2(\alpha^{-1})\, h_2vx_2(\alpha^{-1}).$$

In the last two formulas $x_1(\alpha, \beta) \neq 1$ and $x_2(\alpha) \neq 1$ and h_1 and h_2 denote respectively the diagonal matrices with diagonal entries $(1, \beta, \beta, \beta, 1)$ and $(\alpha, \alpha^{-1}, 1, \bar{\alpha}, \bar{\alpha}^{-1})$.

(3.4) Each element of L has a unique expression in the form $g = xhwy$, where $x \in S$, $h \in \langle e_1, e_2\rangle$, $w \in \langle u, v\rangle$, and $y \in \langle X_i \mid X_i \not\leqslant S^w\rangle$.

(3.5) Let $K_1 = \langle X_1, u\rangle$, $K_2 = \langle X_2, v\rangle$, and $K_3 = \langle Z(X_1)\, X_2X_3Z(X_4), u, v\rangle$. Then $K_1 = X_1\langle e_1\rangle \cup X_1\langle e_1\rangle\, uX_1 \cong SU(3, 2)$ with $Z(K_1) = \langle e_1\rangle$, $K_2 = X_2\langle e_1e_2\rangle \cup X_2\langle e_1e_2\rangle\, vX_2 \cong SL(2, 4)$, and $K_3 \cong PSU(4, 2)$. We remark that (3.3) gives

$$e_1 = x_1(\bar{\omega}, \omega)\, ux_1(1, \omega)\, ux_1(\omega, \omega)u.$$

(3.6) $C_L(e_1) = \langle e_2\rangle K_1 \times \langle Z(X_4), vuv\rangle$ with $\langle Z(X_4), vuv\rangle \cong SL(2, 2)$ and $C_L(e_2) = \langle e_2\rangle \times K_3$. In particular, $C_S(e_1) = X_1Z(X_4)$ and $C_S(e_2) = Z(X_1)\, X_2X_3Z(X_4)$.

(3.7) Define $A_1 = X_2X_3X_4$, $A_2 = X_1X_3X_4$, and $F = Z(X_1)\, X_3Z(X_4)$. Then the following conditions hold.

(1) $Z(S) = S'' = Z(X_4), Z_2(S) = X_3Z(S), Z_3(S) = S' = X_4F, A_1' = Z(S)$, $A_2' = Z(A_2) = \Omega_1(A_2) = F$.

(2) $J_e(S) = F$ and $J_r(S) = A_2$. In particular, $m(S) = 4$.

(3) $C_S(S') = F$ and $C_S(Z_2(S)) = C_S(F) = A_2$.

(4) $\mathscr{E}^*(S/F) = \{A_1F/F, A_2/F\}$ and $\mathscr{E}^*(S/Z(S)) = \{A_1/Z(S), S'/Z(S)\}$.

(5) $A_1 \cong Q_8 * D_8 * D_8$; an extra-special group of minus type of order 2^7.

Proof. (1) and (3) can be easily verified. Since $[X_2, X_4] = [X_3, X_4] = 1$ and $X_2X_3Z(X_4)$ is isomorphic to $D_8 * D_8$, (5) holds. For (4), notice that $\Omega_1(S/Z(S)) = A_1F/Z(S)$ and $\mathscr{E}^*(S/Z(S)) = \mathscr{E}^*(A_1F/Z(S))$. Let B be an elementary abelian subgroup of S of order at least 16. Then B is contained in A_1 or A_2 by (4). Since $m(A_1) = 3$ and $\Omega_1(A_2) = F$, we have $B = F$ and so $J_e(S) = F$. Hence, if C is an abelian subgroup of S of rank 4 then $\Omega_1(C) = F$ and $C \leqslant C_S(F) = A_2$. Thus (2) holds.

(3.8) (1) $N_L(S) = N_L(Z_2(S)) = N_L(S')$.

(2) $N_L(A_1) = N_L(Z(S)) = \langle e_2\rangle K_1A_1$, $\langle e_2\rangle K_1$ is isomorphic to $GU(3, 2)$, and K_1 acts irreducibly on $A_1/Z(S)$.

(3) $N_L(A_2) = N_L(F) = \langle e_2\rangle K_2A_2$, A_2/F is a natural module for $K_2 \cong SL(2, 4)$, and F is a natural module for $K_2 \cong A_5$. Thus $F^{\#}$ decomposes, under the action of K_2, into two orbits of length 5 and 10.

For convenience we set $x_i(\Gamma_0) = \{x_i(\alpha) \mid \alpha \in \Gamma_0\}$ and $x_i(0, \Gamma_0) = \{x_i(0, \beta) \mid \beta \in \Gamma_0\}$. Then $Z(X_i) = x_i(0, \Gamma_0)$, $i = 1, 4$.

(3.9) The group L has precisely two conjugacy classes of involutions and we can choose $x_4(0, 1)$ and $x_3(1)$ as their representatives. Moreover, $C_L(x_4(0, 1)) = N_L(A_1)$ and $C_L(x_3(1)) = \langle e_2\rangle\langle x_2(\Gamma_0), v\rangle A_2$.

Proof. See Thomas [18].

Let $A = \mathrm{Aut}(L)$. Then A is a semidirect product of L and $\langle f\rangle$, where f is the field automorphism of L induced by the automorphism: $\alpha \mapsto \bar{\alpha}$ of Γ (see Steinberg [16]). Thus $x_i(\alpha, \beta)^f = x_i(\bar{\alpha}, \bar{\beta})$, $x_i(\alpha)^f = x_i(\bar{\alpha})$, $(e_j)^f = e_j^{-1}$ for $j = 1, 2$, and f centralizes u and v.

(3.10) (1) Every involution in $\langle f\rangle S - S$ is conjugate to f by an element of S.

(2) $C_S(f) = x_1(0, \Gamma_0)\, x_2(\Gamma_0)\, x_3(\Gamma_0)\, x_4(0, \Gamma_0)$ and $C_L(f) = \langle C_S(f), u, v\rangle \cong Sp(4, 2)$. In addition, $C_S(f)$ is a Sylow 2-subgroup of $C_L(f)$ and $\mathscr{E}^*(C_S(f)) = \{C_{A_1}(f), C_{A_2}(f)\}$.

(3) $m(\langle f\rangle S) = 4$. Moreover, if E is an elementary abelian subgroup of $\langle f\rangle S$ of order 16 then either $E = F$ or $C_{\langle f\rangle S}(E) = E$.

(4) $x^L \cap xC_L(x)^{(\infty)} \neq \{x\}$ for every involution $x \in A - L$.

Proof. (2) can be verified by direct computations. We also have that the number of elements of S which are inverted by f is $|S : C_S(f)|$. Thus (1) holds. Let E be an elementary abelian subgroup of $\langle f\rangle S$ of order at least 16. If $E \nleqslant S$, then by (1) we may assume that f lies in E and so $E \leqslant \langle f\rangle\, C_S(f)$. Since $|C_{A_i}(f)| = 8$, it then follows from (2) that $E = \langle f\rangle\, C_{A_i}(f)$ for $i = 1$ or 2 and $C_{\langle f\rangle S}(E) = E$. This proves (3). For (4), we may assume that $x = f$. Since $C_L(f)^{(\infty)} = C_L(f)'$, $[x_1(0, 1), x_2(1)] = x_3(1)\, x_4(0, 1)$ lies in $C_L(f)^{(\infty)}$. Put $y = x_3(\omega)\, x_4(1, \omega)$. Then we have $f^y = x_3(1)\, x_4(0, 1)f$. Thus (4) holds.

(3.11) Set $X_j^* = \{x_j(\alpha, \beta) \mid \alpha \in \Gamma_0, \beta \in \Gamma, \alpha\bar{\alpha} = \beta + \bar{\beta}\}$, $j = 1, 4$. Then $|X_j^*| = 4$ and the following conditions hold.

(1) $N_L(C_{A_1}(f)) = \langle e_2\rangle\langle Z(X_1), u\rangle\, A_1$ and $N_L(\langle f\rangle\, C_{A_1}(f)) = \langle Z(X_1), u\rangle X_2 X_3 X_4^*$ with $\langle Z(X_1), u\rangle \cong SL(2, 2)$.

(2) $N_L(C_{A_2}(f)) = \langle e_2\rangle\langle x_2(\Gamma_0), v\rangle\, A_2$ and $N_L(\langle f\rangle\, C_{A_2}(f)) = \langle x_2(\Gamma_0), v\rangle X_1^* X_3 X_4^*$ with $\langle x_2(\Gamma_0), v\rangle \cong SL(2, 2)$.

(3) $C_A(\langle f\rangle\, C_{A_i}(f)) = \langle f\rangle\, C_{A_i}(f)$ and $N_A(\langle f\rangle\, C_{A_i}(f))/\langle f\rangle\, C_{A_i}(f)$ is a split extension of an elementary abelian group of order 8 by $SL(2, 2)$, for $i = 1, 2$.

Proof. Let $B_i = C_{A_i}(f)$. Using (3.3) and (3.4) we have $N_L(B_1) = \langle e_2\rangle\langle Z(X_1), u\rangle\, A_1$ and $N_L(B_2) = \langle e_2\rangle\langle x_2(\Gamma_0), v\rangle\, A_2$. Since $(X_j/Z(X_j)) \cap C(f) = X_j^*/Z(X_j)$, $j = 1, 4$, and $N_L(\langle f\rangle\, B_i) = \{x \in N_L(B_i) \mid x^{-1}x^f \in B_i\}$, (1) and (2) hold. Finally, (3.10) shows that B_i is a self-centralizing subgroup of $C_L(f)$. Thus (3) holds.

(3.12) (1) $C_A(e_i) = C_L(e_i)$, $i = 1, 2$.

(2) $C_A(F) = \langle e_2\rangle\, A_2$, $C_A(A_1/Z(S)) = A_1$, $C_A(A_2/F) = A_2$, $C_A(S'/F) = \langle e_1{}^2 e_2\rangle S$, and $C_A(S/S') = S$.

(3) $O_2(N_A(S)) = S$ and $O_2(N_A(A_i)) = A_i$, $i = 1, 2$.

Proof. These assertions can be verified by direct computations.

4. 2-Local Subgroups of G

Henceforth we shall assume the following hypothesis.

(4.1) *Hypothesis.* G is a group which contains a standard subgroup L isomorphic to $PSU(5, 2)$. Furthermore, $C(L)$ has a cyclic Sylow 2-subgroup and $LO(G) \ntrianglelefteq G$.

The symbols defined in Section 3 for various objects of the group $PSU(5, 2)$ will retain their meaning for the remainder of the paper. Thus S is a Sylow 2-subgroup of L.

Let t be an involution of $C(L)$ and set $H = C(t)$. Then L is a normal subgroup of H. Let R be a Sylow 2-subgroup of $LC_H(L)$ with $S \leqslant R$. We begin by studying the fusion of the involution t.

(4.2) *The following conditions hold.*

(1) $\langle t\rangle \in \mathrm{Syl}_2(C(L))$. *In particular*, $C_H(L) = \langle t\rangle\, O(H)$.

(2) $t^{N(F\langle t\rangle)} = Ft$.

Proof. As $t \notin Z^*(G)$ by Hypothesis (4.1), there exists an element $g \in G$ such that $t \neq t^g \in H$ by the Z^*-theorem [5]. If $t^g \in LC_H(L)$, then $t^g \in L\langle t\rangle$ since $C_H(L)$ has cyclic Sylow 2-subgroups. If $t^g \notin LC_H(L)$, then (3.10)(4) implies that $(t^g)^L \cap t^g C_L(t^g)^{(\infty)} \neq \{t^g\}$. Conjugating by g^{-1} we have $t^G \cap tH^{(\infty)} \neq \{t\}$, and so $t^G \cap Lt \neq \{t\}$ since $H^{(\infty)} = L$. Thus in any case $t^G \cap F\langle t\rangle \neq \{t\}$ by (3.9).

Let Q be a Sylow 2-subgroup of H containing R and set $T = C_R(L)$. Let B be a subgroup of Q such that

(i) $\Omega_1(Z(B))$ is elementary abelian of order 2^5 and

(ii) $B/Z(B)$ is noncyclic.

If $E = \Omega_1(Z(B))$ is not contained in R, then $C_{\bar{Q}}(\bar{E}) = \bar{E}$ by (3.10)(3), where $\bar{Q}$ denotes Q/T. This implies that $C_Q(E) \leqslant ET$ and $C_Q(E)/E$ is cyclic, contrary to (ii). Hence $E \leqslant R$. Thus $E = F\langle t\rangle$ by (3.7) and so B is a subgroup of $C_Q(F\langle t\rangle) = A_2T$ by (3.12). It then follows that A_2T is weakly closed in Q with respect to G, since $\Omega_1(Z(A_2T)) = F\langle t\rangle$ and A_2T satisfies (i) and (ii). Therefore, (2.1) implies that each element of $t^G \cap F\langle t\rangle$ is conjugate to t by an element of $N(A_2T)$. If $|\,T\,| > 2$ then $N(A_2T) \leqslant H$ since $Z(A_2T) = F \times T$ and T is cyclic with $t \in T$. This contradicts $t^G \cap F\langle t\rangle \neq \{t\}$. Thus $T = \langle t\rangle$ and (1) holds. In particular, $|\,H/LO(H)| \leqslant 4$ and so $t \notin H'$. On the other hand, $x \in C_L(x)'$ for every involution x of L. Thus $t^G \cap L = \varnothing$ and we obtain $t^G \cap F\langle t\rangle = t^{N(F\langle t\rangle)} \leqslant Ft$.

Set $X = N(F\langle t\rangle)$ and $\Delta = t^X$. By (3.8), $F^\#$ decomposes under the action of $N_L(F)$ into two orbits of length 5 and 10. Hence $|\,\Delta\,| = 6$, 11, or 16 by the above. As $X/C(F\langle t\rangle)$ is isomorphic to a subgroup of $\mathrm{Aut}(F\langle t\rangle) \cong GL(5, 2)$ and 11 does not divide the order of $GL(5, 2)$, $|\,\Delta\,| \neq 11$. Assume $|\,\Delta\,| = 6$. Since $N_L(F)$ acts irreducibly on F, Δ generates $F\langle t\rangle$ and $C(\Delta) = C(F\langle t\rangle)$. Set $\bar{X} = X/C(F\langle t\rangle)$ and consider the permutation group $(\bar{X}, \Delta)$. By (3.12)(2) we have $O_2(C(F\langle t\rangle)) = A_2\langle t\rangle$. Set $X_0 = C_X(A_2\langle t\rangle/F\langle t\rangle)$. Then X/X_0

is isomorphic to a subgroup of $GL(4, 2)$ since $A_2\langle t\rangle/F\langle t\rangle$ is elementary abelian of order 16, and so $|X/X_0|_3 \leqslant 9$. Moreover, $X_0 \cap N_H(F) = C_H(A_2/F) = A_2C_H(L)$ by (3.12) and $|X : A_2C_H(L)| = |\Delta| \cdot |N_H(F) : A_2C_H(L)|$ is divisible by 3^3 by (3.8). Hence $\bar{X} \rhd \bar{X}_0 \neq 1$ and $\bar{X}_0 \cap \overline{N_H(F)} = 1$. But this is impossible since the permutation group $(\bar{X}, \Delta)$ is 2-transitive of degree 6 and $\overline{N_H(F)}$ is the stabilizer of a point t. Thus $|\Delta| = 16$ and (2) holds.

(4.3) DEFINITION. Let $C_0 = O_2(N(F\langle t\rangle))$.

(4.4) *The following conditions hold.*

(1) $N(F\langle t\rangle) = N(A_2\langle t\rangle) = N_H(A_2)\,C_0 \leqslant N(A_2)$ *and* $H \cap C_0 = A_2\langle t\rangle$.

(2) $C_0/A_2\langle t\rangle \simeq F$ *as* $N_H(A_2)$*-modules.*

(3) $[C_0, O(H)] = 1$.

Proof. Let $X = N(F\langle t\rangle)$ and $\bar{X} = X/C(F\langle t\rangle)$. By (3.8)(3) and (4.2) we have a permutation group $(\bar{X}, Ft)$ which is primitive rank 3 with subdegree 1, 5, and 10. Moreover, $|\bar{X}| = 2^4\,|N_H(F)/C_H(F)|$ is equal to $2^6 \cdot 3 \cdot 5$ or $2^7 \cdot 3 \cdot 5$. Note that (4.2) implies $X \rhd F$. Set $U = C_X(F)$. If $U = C(F\langle t\rangle)$, then $\bar{X}$ is isomorphic to a subgroup of $\mathrm{Aut}(F) \simeq GL(4, 2)$. But $GL(4, 2)$ does not have a subgroup of order $2^6 \cdot 3 \cdot 5$ or $2^7 \cdot 3 \cdot 5$. In fact, a Sylow 2-subgroup of $GL(4, 2)$ has order 2^6 and it is a Borel subgroup, and furthermore each maximal parabolic subgroup of $GL(4, 2)$ has order $2^6 \cdot 3^2$ or $2^6 \cdot 3 \cdot 7$. Hence $\bar{X} \rhd \bar{U} \neq 1$. As $\overline{N_H(F)} \cap \bar{U} = 1$, it then follows that $\bar{U}$ is a regular normal subgroup of the permutation group $(\bar{X}, Ft)$. In particular, $X = N_H(F)U$. Now, the map defined by $\bar{x} \mapsto [\bar{x}, t] = t^{\bar{x}}t$ for $x \in U$ is a $N_H(F)$-isomorphism between $\bar{U}$ and F. By (3.12) and (4.2), $C(F\langle t\rangle) = C_L(F)\,C_H(L)$ is 2-closed with $A_2\langle t\rangle$ the Sylow 2-subgroup. Hence $X = N(A_2\langle t\rangle)$. In addition, $C(F\langle t\rangle)' = A_2 \times O(H)'$ since $[A_2, e_2] = A_2$. Thus A_2 is normal in X.

Let $\tilde{X} = X/A_2\langle t\rangle$ and $V = C_X(O(\tilde{U}))$. Then $\widetilde{C(F\langle t\rangle)} = O(\tilde{U})$ and $V \cap N_H(F) \geqslant K_2$. Since K_2 acts irreducibly on $\bar{U} \simeq F$ and $X \rhd V$, it follows that the subgroup $\bar{U}$ is contained in $\bar{V}$. Hence $U = (V \cap U)\,C(F\langle t\rangle)$ and $\tilde{U} = C_{\tilde{U}}(O(\tilde{U}))\,O(\tilde{U})$. Thus $\tilde{U} = O_2(\tilde{U})\,O(\tilde{U})$. As $O_2(N_H(F)) = A_2\langle t\rangle$ by (3.12) and $X = N_H(F)U$, we conclude that $C_0 = O_2(U)$ and $U = C(F\langle t\rangle)\,C_0$. Thus (1) and (2) hold. Finally, $O(H)$ stabilizes the series $C_0 > A_2\langle t\rangle > 1$ and (3) holds.

(4.5) DEFINITION. Let $D_0 = [C_0, N_L(A_2)]$, $Y = Z(D_0)$, and $K = E(C(e_2))$.

(4.6) *The following conditions hold.*

(1) $C_0 = D_0\langle t\rangle > D_0 = A_2Y$ *and* $C_Y(t) = F$.

(2) Y *is an elementary abelian subgroup of* K *of order* 2^8.

(3) *$C_K(t) = K_3$ and $K \cong PSL(4, 4)$ or $PSU(4, 2) \times PSU(4, 2)$. In the former case t acts on K as a graph-field automorphism and in the latter case t interchanges the components of K.*

Proof. Recall that $K_3 = \langle X_2F, u, v\rangle \cong PSU(4, 2)$. Let $B = C(e_2)$. Then $C_B(t) = \langle e_2\rangle K_3C_H(L)$ by (3.6) and (3.12). Thus K_3 is a standard subgroup of B and $\langle t\rangle \in \mathrm{Syl}_2(C_B(K_3))$, so that a result of Gomi [8] applies. Since $C_B(t) \leqslant K_3C_B(K_3)$, (4.5) of Lemma (6E) and Lemma (6F) in [8] show that Case (3) of the main theorem of [8] does not occur in B. Let $U = C_{C_0}(e_2)$. Then $C_0 = A_2U$ and $A_2 \cap U = F$ by (4.4) and so $t^U = Ft$ by (4.2). Now $X_2F \in \mathrm{Syl}_2(K_3)$ and $F = J_r(X_2F)$, so $F\langle t\rangle$ corresponds to the subgroup denoted by B_2 in [8]. As $N_B(F\langle t\rangle) \rhd U$, Case (1) of the main theorem of [8] does not occur in B. Thus (3) holds. Moreover, Lemma (6B) of [8] shows that $O_2(N_B(F\langle t\rangle)) = U$. It then follows from Lemmas (6D) and (6G) of [8] that $N_B(F\langle t\rangle)$ has a normal elementary abelian subgroup E_2 such that $U = E_2\langle t\rangle > E_2 > F$. As $A_2 \lhd N(F\langle t\rangle)$, A_2E_2 is then a subgroup normalized by $N_B(F\langle t\rangle)$ and $C_0 = A_2U = A_2E_2\langle t\rangle$. Now $K_2 \leqslant B \cap N_H(A_2)$ and $A_2E_2/A_2 \cong F$ as K_2-modules by (4.4)(2). Hence $A_2E_2 = [A_2E_2, K_2] = [C_0, K_2]$. Since $N_L(A_2) = \langle e_2\rangle K_2A_2$, we conclude that $A_2E_2 = D_0$.

We wish to show that $E_2 = Y$. Since K_2 acts irreducibly on A_2/F and $A_2 \lhd D_0$, A_2/F is contained in $Z(D_0/F)$ and so $E_2 \lhd D_0$. In addition, $C_{D_0}(e_2) = E_2$ since $C_{A_2}(e_2) = F = A_2 \cap E_2$. Thus $[D_0, e_2]$ centralizes E_2 by [20, Lemma (2.4)]. As $D_0 = [D_0, e_2]\, E_2$ and $Z(A_2) = F$, we conclude that $Y = E_2$. Finally, $Y = [Y, K_2] \leqslant K$. The proof is complete.

(4.7) $N(Y\langle t\rangle) = N_H(A_2)Y$.

Proof. (4.6) implies that $\mathscr{E}^*(Y\langle t\rangle) = \{Y, F\langle t\rangle\}$. Hence the assertion is a consequence of (4.4).

(4.8) Definition. Let $W = C_Y(S)$.

(4.9) $N(Z(R)) = N(A_1\langle t\rangle) = N(W\langle t\rangle) = N_H(A_1)W \leqslant N(W)$, $C_W(t) = Z(S)$, *and* $|W| = 4$. *Furthermore, W is the center of a Sylow 2-subgroup of K which contains X_2Y.*

Proof. Note that $W = C_Y(X_2)$ since $[Y, A_2] = 1$. By (4.6) we have that $J_r(Q) = Y$ and $Z(Q) = C_Y(Q)$ for each Sylow 2-subgroup Q of K containing X_2Y. Hence $W \geqslant Z(Q)$ and $|W| \geqslant 4$. Also, $C_W(t) = Z(S)$ since $C_Y(t) = F$. Moreover, $Y/F \cong F$ as $N_H(A_2)$-modules by (4.4)(2) and so $|C_{Y/F}(X_2)| = |C_F(X_2)|$. Thus $|W| = 4$, and $t^W = Z(S)t$ since $t^Y = Ft$.

As the 2-rank of H is 5, $t^G \cap Y = \varnothing$. Let $X = N(Z(R))$. Then $X \geqslant W$ and $t^X = Z(S)t$, so that $N_H(A_1) = N_H(Z(S))$ is a normal subgroup of X of index 2 and $X = N_H(A_1)W$. As $Z(A_1\langle t\rangle) = Z(R)$, we get that $X = N(A_1\langle t\rangle)$.

Since $C_H(A_1) = Z(S)\,C_H(L)$ by (3.12)(2), $C_X(A_1) = C_H(L)W$. This implies $X \rhd W\langle t\rangle$ since $C_H(L)W = O(H) \times W\langle t\rangle$ by (4.4)(3) and $X \rhd A_1$. As $\mathscr{E}^*(W\langle t\rangle) = \{W, Z(R)\}$ and $t^G \cap W = \varnothing$, we have $X = N(W\langle t\rangle) \rhd W$.

(4.10) Definition. Set $D_1 = C(W) \cap O_2(N(A_1W\langle t\rangle) \cap N(W))$. Let V_2 be the preimage of $C_{Y/W}(S)$ in Y.

(4.11) *The following conditions hold.*

(1) $N(A_1W\langle t\rangle) \cap N(W) = N_H(A_1)\,D_1$, $C_{D_1}(t) = A_1$, $[D_1, O(H)] = 1$, *and D_1 acts transitively on* $A_1W\langle t\rangle/W - A_1W/W$.

(2) $D_1/A_1W \cong A_1/Z(S)$ *as* $N_H(A_1)$*-modules.*

(3) $t^{V_2} = Z_2(S)t$, $C_{V_2}(t) = Z_2(S)$, *and* $V_2/X_3W \cong Z_2(S)/Z(S)$ *as* $N_H(S)$*-modules.*

(4) $A_1V_2 = D_1 \cap N_H(S)Y = N_{D_1}(X_3W\langle t\rangle)$.

Proof. Put $\overline{N(W)} = N(W)/W$. We first prove (3). Let Q be a Sylow 2-subgroup of K containing X_2Y. Then by (4.6) and (4.9), $J_r(Q) = Y$, $Z(Q) = W$, and $Z(\bar{Q}) = C_{\bar{Y}}(Q)$. Note that $\bar{V}_2 = C_{\bar{Y}}(X_2)$ since $[Y, A_2] = 1$, and so $\bar{V}_2 \geqslant Z(\bar{Q})$. Since $t^Y = Ft$ and $C_Y(\bar{t}) = N_Y(W\langle t\rangle) = FW$, the map defined by $\bar{F}\bar{x} \mapsto [\bar{x}, \bar{t}]$ for $x \in Y$ is a $N_H(S)$-isomorphism of $\bar{Y}/\bar{F}$ onto $\bar{F}$. Thus $\bar{Y}/\bar{F} \cong F/Z(S)$ as $N_H(S)$-modules. Since $(F/Z(S)) \cap C(X_2) = Z_2(S)/Z(S)$ has order 4, $|\bar{V}_2| \leqslant 16$. Thus $\bar{V}_2 = Z(\bar{Q})$. This implies that $\bar{t}^{V_2} = \overline{Z_2(S)}\,\bar{t}$. In particular, V_2 normalizes $Z_2(S)\,W\langle t\rangle = X_3W\langle t\rangle$ and $t^{V_2} \leqslant Ft \cap X_3W\langle t\rangle = Z_2(S)t$. Moreover, $[C_{V_2}(t), S] \leqslant W \cap S = Z(S)$ and so $C_{V_2}(t) \leqslant Z_2(S)$. Since $|V_2| = 2^6$, it then follows that $t^{V_2} = Z_2(S)t$ and $C_{V_2}(t) = Z_2(S)$. Now, the map defined by $\bar{x} \mapsto [\bar{x}, \bar{t}]$ for $x \in V_2$ induces a $N_H(S)$-isomorphism of $\bar{V}_2/\overline{Z_2(S)}$ onto $\overline{Z_2(S)}$. Thus (3) holds.

Set $X = N(A_1W\langle t\rangle) \cap N(W)$ and $U = C_X(\bar{A}_1)$. Then $A_1W = A_1W\langle t\rangle \cap C(W) \lhd X$, so $\bar{t}^X \leqslant \bar{A}_1\bar{t}$ and $U \lhd X$. Consider the map defined by $x \mapsto [x, \bar{t}]$ for $x \in U$. This is a $N_H(A_1)$-homomorphism of U into $\bar{A}_1$. By (4.9) $C_X(\bar{t}) = N_H(A_1)W$, so (3.12) implies that $C_U(\bar{t}) = A_1WC_H(L)$. Since V_2 normalizes $X_3W\langle t\rangle$ and centralizes $\bar{S}$, $V_2 \leqslant U$. Moreover, $V_2 \nleqslant C_U(\bar{t})$ and $N_L(A_1)$ acts irreducibly on $\bar{A}_1 \cong A_1/Z(S)$, so the above homomorphism is surjective. Therefore, $\bar{t}^U = \bar{A}_1\bar{t} = \bar{t}^X$, $X = C_X(\bar{t})U = N_H(A_1)U$, and $U/C_U(\bar{t}) \cong A_1/Z(S)$ as $N_H(A_1)$-modules. As $C_U(O(H))\,O(H)$ contains $C_U(\bar{t})\,V_2$, the irreducible action of $N_L(A_1)$ on $U/C_U(\bar{t})$ yields that $U = C_U(O(H))\,O(H)$. Thus $U = O_2(U) \times O(H)$. Hence $X = N_H(A_1)\,O_2(U)$, and as $O_2(N_H(A_1)) = A_1\langle t\rangle$ by (3.12), we have $O_2(X) = O_2(U) \geqslant A_1V_2\langle t\rangle$. Since $N_L(A_1)$ acts irreducibly on $O_2(X)/A_1W\langle t\rangle$, this implies that $O_2(X) = D_1\langle t\rangle$. Thus (1) and (2) hold.

For an element $x \in D_1 \cap N_H(S)Y$ we have $\bar{t}^x \in \bar{A}_1\bar{t} \cap \bar{F}\bar{t} = \bar{t}^{V_2}$. As $C_{D_1}(\bar{t}) = A_1W$, it then follows that $D_1 \cap N_H(S)Y = A_1V_2$. We also have $N_{D_1}(\overline{X_3\langle t\rangle}) = A_1V_2$, proving (4).

(4.12) DEFINITION. Let $R_2 = RV_2$, $R_3 = O_2(N(R_2))$, and $D_3 = N_{D_1}(R_2)$.

(4.13) *The following conditions hold.*

(1) $R_2' = S'W$.

(2) $N(R_2) = N_H(S)\, R_3$ *and* $C_{R_3}(t) = R$.

(3) $R_3/R_2 = RY/R_2 \times RD_3/R_2$. *In particular,* R_3/R_2 *is elementary abelian of order* 8.

(4) $D_3/A_1V_2 \cong X_3X_4/Z_2(S)$ *and* $D_1/D_3 \cong A_1/X_3X_4$ *as* $N_H(S)$*-modules.*

(5) $D_3 = N_{D_1}(X_3X_4W\langle t\rangle)$. *Moreover,* D_3 *acts transitively on* $X_3X_4W\langle t\rangle/W - X_3X_4W/W$.

Proof. As V_2 normalizes SW and $F\langle t\rangle$, RW is a normal subgroup of R_2 of index 4. Since $R_2/S'V_2$ is abelian, we have $R_2' \leqslant RW \cap S'V_2 = S'W$ and as $|\, S'W : S' \,| = 2$, R_2' is equal to $S'W$ or S'. If $R_2' = S'$ then $R_2 \rhd R$. But V_2 does not normalize $Z(R)$ by (4.9). Hence (1) holds. Then $C_{R_2}(R_2') = FV_2$ since $C_S(S') = F$ and $[V_2, S'] \leqslant [Y, A_2] = 1$. Moreover, $R_2/FV_2 \cong R/F$ implies that $Z(R_2 \bmod FV_2) = S'V_2\langle t\rangle$. Thus

$$N(R_2) \leqslant N(FV_2) \cap N(S'V_2\langle t\rangle) \cap N(S'V_2).$$

Let $\widetilde{N(FV_2)} = N(FV_2)/FV_2$, $X = N(R_2)$, and $U = C_X(\tilde{S}')$. Then $\tilde{t}^X \leqslant \tilde{S}'\tilde{t}$ and $U \lhd X$. As $\mathscr{E}^*(FV_2\langle t\rangle) = \{FV_2, F\langle t\rangle\}$, $N(FV_2\langle t\rangle) \leqslant N_H(A_2)Y$ by (4.4). Also, Y normalizes R_2 since $|\, YR : R_2 \,| = 2$. Thus $C_X(\tilde{t}) = N_H(S)Y$. Define a $N_H(S)$-homomorphism from U into $\tilde{S}'$ by $x \mapsto [x, \tilde{t}]$. (4.11)(4) shows that $C_{D_3}(\tilde{t}) = A_1V_2$. Furthermore, $A_1V_2 \neq D_3$ since $R_2 < RD_1$ and $R_2 \cap D_1 = A_1V_2$. Since $N_L(S)$ acts irreducibly on $\tilde{S}' \cong S'/F$ and $\tilde{S}'$ is a normal subgroup of $\tilde{D}_3$, we have $D_3 \leqslant U$ and the above homomorphism maps D_3 onto $\tilde{S}'$. Thus $\tilde{t}^{D_3} = \tilde{S}'\tilde{t}$ and $D_3/A_1V_2 \cong S'/F \cong X_3X_4/Z_2(S)$ as $N_H(S)$-modules. In particular, $\tilde{t}^X = \tilde{S}'\tilde{t}$ and $X = C_X(\tilde{t})\, D_3$. Now $C_U(\tilde{t}) = C_H(S'/F)Y$ is 2-closed with $RY = O_2(C_U(\tilde{t}))$ by (3.12)(2). In addition, $U = C_U(\tilde{t})\, D_3 \rhd C_U(\tilde{t})$ since $C_U(\tilde{t})$ is the kernel of the above homomorphism. Thus $O_2(U) = RYD_3 \rhd RY$. As $O_2(N_H(S)) = R$ by (3.12)(3), we conclude that $R_3 = RYD_3$. Since $RD_3 \cap RY = R(D_3 \cap RY) = R_2$ by (4.11)(4), (2) and (3) hold.

As $D_1 \leqslant N(A_1W\langle t\rangle)$, D_3 normalizes $S'V_2\langle t\rangle \cap A_1W\langle t\rangle = X_3X_4W\langle t\rangle$. Let $\overline{N(W)} = N(W)/W$. By (4.9) and (4.11), $C_{D_1}(\bar{t}) = A_1W$ and D_1 acts transitively on $\bar{A}_1\bar{t}$. Also, D_1 centralizes $\bar{A}_1$ since $A_1W \lhd D_1$ and $N_L(A_1)$ acts irreducibly on $\bar{A}_1$. Now $|\, D_3/A_1V_2 \,| = |\, A_1V_2/A_1W \,| = 4$, so D_3 acts transitively on $\bar{X}_3\bar{X}_4\bar{t}$. Thus $D_3 = N_{D_1}(X_3X_4W\langle t\rangle)$. Furthermore, the $N_H(A_1)$-isomorphism of D_1/A_1W onto $\bar{A}_1$ defined by $A_1Wx \mapsto [x, \bar{t}]$ for $x \in D_1$ maps D_3 onto $\bar{X}_3\bar{X}_4$. Hence $D_1/D_3 \cong A_1/X_3X_4$ as $N_H(S)$-modules. The proof is complete.

(4.14) DEFINITION. Let $D_2 = C(Y) \cap O_2(N(C_0))$ and $S_3 = C_{R_3}(W)$.

(4.15) *The following conditions hold.*

(1) $N(C_0) = N_H(A_2)\, D_2 \leqslant N(D_0)$, $C_{D_2}(t) = A_2$, $[D_2, O(H)] = 1$, *and* D_2 *acts transitively on* $C_0/Y - D_0/Y$.

(2) $D_2/D_0 \cong A_2/F$ *as* $N_H(A_2)$*-modules.*

(3) $\mathscr{E}^*(RD_2/C_0) = \{D_2\langle t\rangle/C_0, R_3/C_0\}$. *In particular,* $R_3 \lhd RD_2$.

(4) $R_3 \cap R_3^v \cap D_2 = D_0$.

(5) $R_3 = S_3\langle t\rangle > S_3 = D_0 D_3$.

Proof. (5) is a consequence of (4.13)(3). Since RY/Y is isomorphic to R/F and $|RY : R_2| = 2$, $(RY)' \leqslant S'Y \cap R_2 = S'V_2$. On the other hand, $(RY)'$ contains $S'W$ by (4.13)(1). Hence, $C_S(S') = F$ implies $C_{RY}((RY)') = Y$. Also, $C_{RY}(Y) = D_0$ since $C_S(F) = A_2$. Furthermore, (3.7)(4) implies $\mathscr{E}^*(RY/Y) = \{A_1Y\langle t\rangle/Y, C_0/Y\}$. As $A_1Y\langle t\rangle \ngeqslant D_0 \leqslant C_0$, it follows that $N(RY) \leqslant N(C_0)$. We also have that $N(C_0) \leqslant N(D_0) \leqslant N(Y)$ since $\Omega_1(D_0) = Y$ is the unique elementary abelian subgroup of C_0 of order 2^8 and $D_0 = C_{C_0}(Y)$.

Let $\overline{N(Y)} = N(Y)/Y$ and $X = N(C_0)$. By (4.13), $R_3 \leqslant N(RY)$ and $\bar{t}^{R_3} = \bar{X}_4\bar{t}$ Moreover, $N_L(A_2)$ acts transitively on the nonidentity elements of $\bar{D}_0$, for $\bar{D}_0 = \bar{A}_2 \cong A_2/F$. Thus $\bar{t}^X = \bar{D}_0\bar{t}$. (4.7) shows that $C_X(\bar{t}) = N_H(A_2)Y$. Hence $C_X(\bar{C}_0) = C_X(\bar{t}) \cap C(\bar{D}_0) = C_H(A_2/F)Y = O(H) \times C_0$ by (3.12) and (4.4)(3). Set $U = C_X(\bar{D}_0)$ and consider the 2-transitive permutation group $(X/C_X(\bar{C}_0), \bar{D}_0\bar{t})$. If $U = C_X(\bar{C}_0)$ then $X/C_X(\bar{C}_0)$ is isomorphic to a subgroup of $\mathrm{Aut}(\bar{D}_0) \cong GL(4,2)$. But $|X/C_X(\bar{C}_0)| = 2^4\,|C_X(\bar{t})/C_X(\bar{C}_0)| = 2^6 \cdot 3^2 \cdot 5$ or $2^7 \cdot 3^2 \cdot 5$, whereas $GL(4,2)$ does not have subgroups of such orders, a contradiction. Hence $U \neq C_X(\bar{C}_0)$. As $C_U(\bar{t}) = C_X(\bar{C}_0)$, it follows that $U/C_X(\bar{C}_0)$ is a regular normal subgroup of the permutation group $(X/C_X(\bar{C}_0), \bar{D}_0\bar{t})$. Thus $X = C_X(\bar{t})U$. Furthermore, the map defined by $x \mapsto [x, \bar{t}]$ for $x \in U$ is a $N_H(A_2)$-homomorphism of U onto $\bar{D}_0$ with kernel $C_X(\bar{C}_0)$. Notice that $O(H) = O(C_X(\bar{C}_0))$ is a normal subgroup of X. Then $O(H)\, C_X(O(H))$ contains U since $C_X(O(H)) \geqslant K_2C_0$ and K_2 acts irreducibly on $U/C_X(\bar{C}_0) \cong A_2/F$. Thus $U = O(H) \times O_2(U)$ and $X = C_X(\bar{t})\, O_2(U)$. As $O_2(C_X(\bar{t})) = C_0$, this implies that $O_2(X) = O_2(U)$. Therefore, setting $C_2 = O_2(X)$ we have

(a) $N(C_0) = N_H(A_2)\, C_2$, $[C_2, O(H)] = 1$, and $C_2/C_0 \cong A_2/F$ as $N_H(A_2)$-modules.

In the proof of (4.13) we have shown that $S'V_2$ is a normal subgroup of R_3. Recall also that $R_3 \leqslant X$. As $N_L(S)$ acts irreducibly on $\bar{S}'$, $C_X(\bar{S}')$ contains R_3. Moreover, as C_2 centralizes $\bar{D}_0$, it follows that $C_X(\bar{S}') = C_H(S'/F)\, C_2$. Now, $C_H(S'/F)\, C_2$ is 2-closed with SC_2 the unique Sylow 2-subgroup by (3.12)(2). Thus $R_3 \leqslant SC_2$. Furthermore, $R_3 \cap C_2 \lhd R_3$ and $SC_0 \cap C_2 \leqslant C_X(\bar{t}) \cap C_2 = C_0$. In addition, $SC_0 = RY \lhd R_3$ by (4.13). Hence $R_3/C_0 = SC_0/C_0 \times (R_3 \cap C_2)/C_0$. Define a $N_H(A_2)$-isomorphism of C_2/C_0 onto $\bar{D}_0 = \bar{A}_2$ by $C_0x \mapsto [x, \bar{t}]$ for $x \in C_2$. As R_3 acts transitively on $\bar{X}_4\bar{t}$ and $\bar{X}_4 = \bar{S}'$, this

$N_H(A_2)$-isomorphism maps $R_3 \cap C_2/C_0$ onto $\bar{S}'$. Here note that $N_H(A_2) \geqslant \langle X_2, v\rangle$. As $S = X_2A_2$ and $SC_0 = X_2C_0$, the fact that $\mathscr{E}^*(S/F) = \{X_2S'/F, A_2/F\}$ implies that

(b) $\mathscr{E}^*(SC_2/C_0) = \{R_3/C_0, C_2/C_0\}$.

Similarly, as $S' \cap S'^v = F$, we have

(c) $R_3 \cap R_3{}^v \cap C_2 = C_0$.

In order to establish (4.15) we argue that $C_2 = D_2\langle t\rangle$. Set $Q = S_3 \cap C_2$. Then $R_3 \cap C_2 = Q\langle t\rangle > D_0$ by (5), and thus $Q/D_0 \cong R_3 \cap C_2/C_0 \cong S'/F$ as $N_L(S)$-modules by the above. Also, $D_0/Y \cong A_2/F$. Hence $C_Q(e_2) = Y$. Then as $Y \lhd Q$, $[Q, e_2]$ centralizes Y by Lemma (2.4) of [20], and $Q = C_Q(e_2)[Q, e_2]$ lies in $C(Y)$. Thus $D_2 \geqslant QQ^v$. Since $Q \cap Q^v = D_0$ by (c), the order consideration gives that $C_2 = D_2\langle t\rangle > D_2 = QQ^v$. Now (4.15) follows from (a), (b), and (c).

(4.16) Definition. Let $M = C_{D_1}(e_2)$ and $V_3 = Z(S_3 \bmod Y)$.

(4.17) *The following conditions hold.*

(1) *MY is a Sylow 2-subgroup of K. Furthermore, $Z(MY) = W$, $Z_2(MY) = V_2$, $J_r(MY) = Y$, and $J_r(MY/W) = M/W$.*

(2) *$D_1' = W$ and $\mathscr{E}^*(D_1\langle t\rangle/W) = \{A_1W\langle t\rangle/W, D_1/W\}$.*

(3) *$D_2' = Y$, $\mathscr{E}^*(D_2\langle t\rangle/Y) = \{C_0/Y, D_2/Y\}$, and $D_2/Y = V_3/Y \times V_3{}^v/Y$.*

(4) *$D_1Y/V_2 = D_1/V_2 \times Y/V_2$.*

(5) *$J_r(D_1Y/W) = D_1/W$.*

(6) *$S_3 = SV_3$, $S \cap V_3 = S'$, and $|V_3| = 2^{12}$.*

(7) *$V_3 = [D_3, e_2]Y = D_1Y \cap D_2$ and $V_3' = W$.*

(8) *$D_3 = X_2(D_1 \cap D_2)$.*

Proof. By (4.11)(2), $(D_1/A_1W) \cap C(e_2)$ has order 16 and e_1 acts fixed-point-freely on D_1/A_1W. Hence $|M| = 2^{10}$ and $C_M(e_1e_2) = W$. As $D_1 \cap Y = V_2$ by (4.11)(4), $|MY| = 2^{12}$. Moreover, $M = [M, e_1e_2]W \leqslant K$ since $e_1e_2 \in K_3$. Notice that $C_S(e_2) = X_2F$ is a Sylow 2-subgroup of K_3 and that $X_2F \leqslant FM \cap X_2Y$. If $K \cong PSU(4, 2) \times PSU(4, 2)$, then t interchanges the components of K and so there exists a unique $\langle t\rangle$-invariant Sylow 2-subgroup Q of K which contains X_2F. Certainly, Q contains FM and X_2Y. Thus we get that $Q = MY$. If $K \cong PSL(4, 4)$, take Q so that $X_2Y \leqslant Q \in \mathrm{Syl}_2(K)$. Then $Z(Q) = W$ by (4.9) and for each Sylow 2-subgroup Q_0 of $C_K(W)$ different from Q we have that $Q \cap Q_0/W$ is elementary abelian. Since FM is a 2-subgroup of $C_K(W)$ containing X_2F and since X_2FW/W is nonabelian, this implies $Q \geqslant FM$. Hence $Q = MY$. Thus in either case MY is a Sylow 2-subgroup of K. In particular, $J_r(MY) = Y$ and $Z(MY) = W$. Let $\overline{N(W)} = N(W)/W$ Then $Z(\overline{MY})$ is a subgroup of $\bar{Y}$ of order 16. As A_2 centralizes Y, $\bar{V}_2 = C_{\bar{Y}}(\bar{X}_2)$ by the definition of V_2. Since $|\bar{V}_2| = 16$, this yields that $Z_2(MY) = V_2$. By (4.13), e_2 centralizes D_1/D_3, so $[D_1, e_2] = [D_3, e_2]$ and $D_1 = [D_3, e_2]M$. Since $D_3 \leqslant SD_2$ and $S \cap D_2 = A_2$ by (4.15), $[D_3, e_2] \leqslant D_2$. Therefore, $Z(\bar{D}_1) \geqslant \bar{V}_2$. Since $Z(\bar{D}_1) \geqslant \bar{A}_1$ and $N_L(A_1)$ acts irreducibly on D_1/A_1W by (4.11)(2), we conclude that $\bar{D}_1$ is elementary abelian. In particular, M/W is elementary abelian of order 2^8. It then follows from the structure of K that $J_r(MY/W) = M/W$, and (1) holds.

As $A_1' = Z(S)$, D_1' is equal to W or $Z(S)$. If $D_1' = Z(S)$, then V_2 normalizes $A_1A_2\langle t\rangle = R$ since $V_2 \leqslant N(A_2\langle t\rangle)$ by (4.4). But $V_2 \nleqslant N(Z(R))$, a contradiction. Thus $D_1' = W$. (4.9) implies that $C_{D_1}(\bar{t}) = N_{D_1}(W\langle t\rangle) = A_1W$. Hence $\mathscr{E}^*(\bar{D}_1\langle\bar{t}\rangle) = \{\bar{A}_1\bar{W}\langle\bar{t}\rangle, \bar{D}_1\}$ and (2) holds.

Since $D_1 = D_3M$ and $D_3 \leqslant SD_2$, D_1 normalizes Y. As $|D_1Y : FD_1| = |FD_1 : D_1| = 2$, we have $[D_1Y, e_1] = [D_1Y, e_1, e_1] \leqslant D_1$. Furthermore, $[\bar{D}_1, \bar{e}_1] = \bar{D}_1$ by (4.11)(2). Thus $\bar{D}_1 = [\bar{D}_1\bar{Y}, \bar{e}_1]$. This implies that D_1 is normal in D_1Y. As $D_1 \cap Y = V_2$ by (4.11)(4), (4) holds.

It follows from (4.13)(4) that $C_{D_3}(e_2) = X_2V_2$. As D_3/V_2 is abelian, we have

$$D_3/V_2 = X_2V_2/V_2 \times [D_3, e_2]\, V_2/V_2 .$$

Let $U = [D_3, e_2]Y$ and $\widetilde{N(Y)} = N(Y)/Y$. Then $\tilde{D}_3 = \tilde{X}_2 \times \tilde{U}$ and $|U| = 2^{12}$ since $D_3 \cap Y = V_2$. By (4.15), S_3 is a subgroup of SD_2 and e_2 centralizes SD_2/D_2, so $U \leqslant S_3 \cap D_2$. Moreover, $S_3 = S(S_3 \cap D_2) = D_3Y(S_3 \cap D_2)$ and $|S_3 \cap D_2 : D_0| = 4$. Therefore, comparing orders we get that $U = D_3Y \cap S_3 \cap D_2 = D_3Y \cap D_2$. Since $D_2 \rhd D_0$ and $N_L(A_2)$ acts irreducibly on $\tilde{D}_0$, $Z(\tilde{D}_2)$ contains $\tilde{D}_0$. As $S_3 = D_0D_3$ and $\tilde{D}_3$ is abelian, it follows that $\tilde{U} \leqslant Z(\tilde{S}_3)$. Furthermore, $|S_3 : SY| = 4$, so that $|Z(\tilde{S}_3)| \leqslant 16$ since the center of $\tilde{S} \cong S/F$ is of order 4. Thus $\tilde{U} = Z(\tilde{S}_3) = \tilde{V}_3$. Hence, we have $V_3 = [D_3, e_2]Y = D_3Y \cap D_2$ and $|V_3| = 2^{12}$. Now, $D_3 = X_2V_3$ and so $S_3 = SV_3$. Thus (6) holds.

Since $D_1 \leqslant N(A_1W\langle t\rangle)$ and $D_2 \leqslant N(C_0)$, $D_1 \cap D_2$ normalizes $A_1W\langle t\rangle \cap C_0 = X_3X_4W\langle t\rangle$. Hence $D_1 \cap D_2 \leqslant D_3$ by (5.13)(5). Then, as $[D_3, e_2]\, V_2 \leqslant D_1 \cap D_2$, (8) holds.

We have $V_3 = (D_3 \cap D_2)Y = (D_1 \cap D_2)Y = D_1Y \cap D_2$. Thus for the proof of (7) it is enough to show that $V_3' = W$. Since V_3 is a central product of $D_1 \cap D_2$ and Y, $Z(S) \leqslant V_3' \leqslant W$. Suppose $V_3' = Z(S)$. Then S' is normal in V_3. Since $C_{A_2}(S') = F$, V_3 then normalizes $C_{C_0}(S') = Y\langle t\rangle$. This, however, conflicts with (4.7). Therefore, $V_3' = W$ and (7) holds.

Let $\bar{D}_1\bar{Y} = D_1Y/W$. Then $\overline{D_1 \cap D_2} \leqslant Z(\bar{D}_1\bar{Y})$. Moreover, $D_1 = (D_1 \cap D_2)M$ by (8) and so $\bar{D}_1\bar{Y}$ is a central product of $\overline{D_1 \cap D_2}$ and $\overline{MY}$. As $J_r(\overline{MY}) = \bar{M}$ by (1), this proves (5).

Since D_2 normalizes S_3 by (4.15)(3), $V_3 \lhd D_2$. Now, (4.15)(4) implies

$V_3 \cap V_3{}^v \leqslant D_0$. Furthermore, $V_3 \cap D_0/Y$ is a subgroup of $Z(SY/Y) = S'Y/Y$. Thus $V_3 \cap V_3{}^v \leqslant S'Y \cap (S'Y)^v = Y$. Comparing orders we see that $D_2/Y = V_3/Y \times V_3{}^v/Y$. In particular, D_2/Y is elementary abelian. Since D_2' contains $A_2' = F$ and $V_3' = W$, we get $D_2' = Y$. By (4.7), $C_{D_2}(\tilde{t}) = N_{D_2}(Y\langle t\rangle) = D_0$ where $\widetilde{N(Y)} = N(Y)/Y$. Therefore, $\mathscr{E}^*(\tilde{D}_2\langle\tilde{t}\rangle) = \{\tilde{C}_0, \tilde{D}_2\}$ and (3) holds.

(4.18) *The following conditions hold.*

(1) $C_{S_3}(S') = Y$.

(2) $N(D_i\langle t\rangle) = N_H(A_i)\, D_i$, $i = 1, 2$.

(3) $N(V_3\langle t\rangle) = N_H(S)\, V_3$.

Proof. Since $W \leqslant (D_1\langle t\rangle)' \leqslant A_1W$ and $N_L(A_1)$ acts irreducibly on A_1W/W, $(D_1\langle t\rangle)'$ is equal to W or A_1W. As $Z(A_1W) = W$, it then follows from (4.17)(2) that $N(D_1\langle t\rangle)$ normalizes both W and $A_1W\langle t\rangle$. Thus $N(D_1\langle t\rangle) = N_H(A_1)\, D_1$ by (4.11). Similarly, $Y \leqslant (D_2\langle t\rangle)' \leqslant D_0$ and $N_L(A_2)$ acts irreducibly on D_0/Y, so that $(D_2\langle t\rangle)'$ is equal to Y or D_0. Since $Z(D_0) = Y$, (4.17)(3) and (4.15) imply that $N(D_2\langle t\rangle) = N_H(A_2)\, D_2$. Let $\widetilde{N(Y)} = N(Y)/Y$. Recall that $C_{S_3}(S')$ contains Y. Suppose (1) is false. Then $\tilde{t}$ centralizes some nonidentity element of $\widetilde{C_{S_3}(S')}$ and so $C_{S_3}(S') \cap N(Y\langle t\rangle) \neq Y$. However, $H \cap S_3 = S$ and $C_S(S') = F$, so that $C_{S_3}(S') \cap N(Y\langle t\rangle) = Y$ by (4.7). This contradiction proves (1).

As $V_3' = W$ and $(Y\langle t\rangle)' = F$, $(V_3\langle t\rangle)'$ contains FW. Let $\overline{N(W)} = N(W)/W$. By (4.13)(5) and (4.17)(8), $D_1 \cap D_2$ acts transitively on $\bar{X}_3\bar{X}_4\bar{t}$. Hence $(V_3\langle t\rangle)'$ contains $X_3X_4FW = S'W$. Furthermore, $R_3' \leqslant R_2 \cap V_3 = S'V_2$, so $(V_3\langle t\rangle)' \leqslant S'V_2$. Thus (1) shows that $V_3\langle t\rangle \cap C((V_3\langle t\rangle)') = Y$. As $H \cap V_3 = S'$, $C_{V_3}(\tilde{t}) = S'Y$ by (4.7) and hence $\mathscr{E}^*(V_3\langle t\rangle/Y) = \{V_3/Y, S'Y\langle t\rangle/Y\}$. Similarly, as $N(W\langle t\rangle) = N_H(A_1)W$, we have $C_{V_3}(\bar{t}) = S'W$ and $\mathscr{E}^*(V_3\langle t\rangle/W) = \{V_3/W, S'W\langle t\rangle/W\}$. Since $V_3' = W$, it follows that $N_H(S)\, V_3 \leqslant N(V_3\langle t\rangle) \leqslant X$, where we set $X = N(S'W\langle t\rangle) \cap N(W)$. Now $S'W = S'W\langle t\rangle \cap C(W)$ is normal in X. Moreover, $|\, V_3 : C_{V_3}(\bar{t})| = 2^5$. Hence V_3 acts transitively on $\bar{S}'\bar{t}$ and $X = C_X(\bar{t})\, V_3$. Since $C_X(\bar{t}) = N_X(W\langle t\rangle) = N_H(S)W$, this proves (3).

(4.19) Definition. Let $P = C(W) \cap O_2(N(R_3))$.

(4.20) $N(R_3) = N_H(S)P \trianglerighteq P = D_1D_2$, $C_P(t) = S$, *and* $P/S_3 \cong S/S'$ *as* $N_H(S)$*-modules.*

Proof. Let $\widetilde{N(Y)} = N(Y)/Y$. Then $\tilde{S}_3$ is a central product of $\tilde{S}$ and $\tilde{V}_3$ with $\tilde{S} \cap \tilde{V}_3 = \tilde{S}'$. By (4.7), $Z(R_3 \bmod Y) \leqslant N_{R_3}(Y\langle t\rangle) = RY$, hence $Z(\tilde{R}_3) = \tilde{S}'$. Now A_1F/F and A_2/F are elements of $\mathscr{A}(S/F)$, so $\tilde{A}_1\tilde{V}_3$ and $\tilde{A}_2\tilde{V}_3$ are elements of $\mathscr{A}(\tilde{S}_3)$ with order 2^6. Let $\tilde{B}$ be an abelian subgroup of $\tilde{R}_3$ not contained in $\tilde{S}_3$.

Then $\tilde{R}_3 = \tilde{S}_3\tilde{B}$ and $|\tilde{B}: \tilde{S}_3 \cap \tilde{B}| = 2$. Since $(\tilde{S}_3 \cap \tilde{B})\tilde{V}_3$ is an abelian subgroup of $\tilde{S}_3$, its order is less than or equal to 2^6. Moreover, $\tilde{B} \cap \tilde{V}_3$ is a subgroup of $Z(\tilde{R}_3)$. Thus $|\tilde{S}_3 \cap \tilde{B}| \leqslant 2^4$ and $|\tilde{B}| \leqslant 2^5$. This implies $\mathscr{A}(\tilde{R}_3) = \mathscr{A}(\tilde{S}_3)$ and $J_0(\tilde{R}_3) = \tilde{S}_3$. As $S'W \leqslant R_3' \leqslant R_2 \cap V_3 = S'V_2$, (4.18)(1) yields that $C_{R_3}(R_3') = Y$. Therefore, we have $N(R_3) \leqslant N(Y) \cap N(S_3) \leqslant N(V_3)$.

Let $\overline{N(V_3)} = N(V_3)/V_3$, $X = N(R_3)$, and $U = C_X(\bar{S}_3)$. Now, $RD_3 < RD_1$ implies $D_3 < N_{D_1}(RD_3)$, so that D_1 in fact normalizes RD_3 since $N_L(S)$ acts irreducibly on D_1/D_3. Hence D_1 normalizes $R_3 = RD_3Y$. By (3.7)(4), $\mathscr{E}^*(S_3/Y) = \{A_1V_3/Y, A_2V_3/Y\}$ and so D_1 normalizes A_1V_3 and A_2V_3 since $A_1V_3 \leqslant D_1Y \not\geqslant A_2V_3$. Likewise D_2 normalizes A_1V_3 and A_2V_3 as well. Hence U contains D_1 and D_2, for $N_L(S)$ acts irreducibly both on $\bar{A}_1$ and on $\bar{A}_2$. Consider the $N_H(S)$-homomorphism of U into $\bar{S}_3$ defined by $x \mapsto [x, \bar{t}]$. By (4.18), $C_X(\bar{t}) = N_H(S)\,R_3$. In addition, $\bar{S}_3 \simeq S/S'$ and $C_H(S/S') = C_H(L)S$ by (3.12)(2). Thus the kernel of this homomorphism is $C_H(L)\,R_3$. (4.11)(1) and (4.15)(1) show that D_i acts transitively on $\bar{A}_i\bar{t}$, $i = 1, 2$. Hence the homomorphism maps D_i onto $\bar{A}_i$ and so is surjective. This implies that $U = C_U(\bar{t})\,D_1D_2$ and $\bar{t}^U = \bar{S}_3\bar{t}$. Since $O(H)$ centralizes D_1 and D_2 and R_3 is a subgroup of RD_2, we have $U = O_2(U) \times O(H)$ with $O_2(U) = D_1D_2R$. Furthermore, $X = C_X(\bar{t})U = N_H(S)\,O_2(U)$. As $O_2(N_H(S)) = R$, we conclude that $O_2(X) = O_2(U)$. Finally, W centralizes D_1 and D_2 but not t, so $P = D_1D_2$. The proof is complete.

(4.21) $Z(P) = W$, $Z_2(P) = V_2$, $Z(P \bmod Y) = V_3$, *and* $C_P(V_2) = D_2$.

Proof. Assume that $C_P(S) > W$. Then t centralizes some nonidentity element of $C_P(S)/W$, so $C_P(S) \cap N(W\langle t\rangle) > W$. This, however, conflicts with (4.9). Therefore, $C_P(S) = W$ and $Z(P) = W$. By (4.11)(3), V_2 acts transitively on $Z_2(S)t$ and $\mathscr{E}^*(V_2\langle t\rangle) = \{V_2, Z_2(S)\langle t\rangle\}$. Setting $X = N(V_2\langle t\rangle)$ we have $t^X = Z_2(S)t$ and $X = C_X(t)\,V_2 = N_H(S)\,V_2$. Now, $Z_2(P)$ contains V_2. Since $C_P(SW/W) \cap X = C_S(SW/W)\,V_2 = Z_2(S)\,V_2 = V_2$, it follows that $C_P(SW/W) = V_2$ and $Z_2(P) = V_2$. Similarly, (4.18)(3) yields that $C_P(SY/Y) \cap N(V_3\langle t\rangle) = C_S(SY/Y)\,V_3 = S'V_3 = V_3$, and hence $Z(P/Y) = V_3/Y$. Also, $C_P(V_2) \cap N(D_2\langle t\rangle) = C_S(Z_2(S))\,D_2 = D_2$ by (4.18)(2) and we have $C_P(V_2) = D_2$. The proof is complete.

(4.22) Definition. Let $N_2 = E(N(D_2) \bmod D_2)$.

(4.23) $N_2 = \langle P, P^v\rangle = \langle MY, v\rangle\,D_2$ *and P is a Sylow 2-subgroup of N_2. Furthermore, N_2/D_2 is isomorphic to $SL(2,4) \times SL(2,4)$ and t interchanges the components of N_2/D_2.*

Proof. By (4.6) and (4.17)(1) we have $N_K(Y)' = \langle MY, v\rangle = \langle M, M^v\rangle Y$, $N_K(Y)'/Y \simeq SL(2,4) \times SL(2,4)$, and t interchanges the components of $N_K(Y)'/Y$. Let $X = N(D_2)$ and $\bar{X} = X/D_2$. Then $C_{\bar{X}}(\bar{t}) = \overline{N_H(A_2)} \simeq$

$N_H(A_2)/A_2$ by (4.18). Hence, $\bar{K}_2$ is a standard subgroup of $\bar{X}$ isomorphic to $SL(2, 4)$ and $\langle \bar{t} \rangle$ is a Sylow 2-subgroup of $C_{\bar{X}}(\bar{K}_2)$. Now $\bar{P}$ is elementary abelian of order 16 and furthermore $SD_2/D_2 \cong S/A_2$ and $P/SD_2 \cong D_1/D_3 \cong S/A_2$ as $N_L(S)$-modules. Since $e_1e_2 \in K_2 \cap N_L(S)$, we get that $\bar{P} = [\bar{P}, \bar{e}_1\bar{e}_2] \leqslant \langle \bar{K}_2^{\bar{X}} \rangle$. Then a result of Griess, Mason, and Seitz [12], together with Lemmas (2.9) and (2.10) of [20], shows that $\bar{N}_2 = \langle \bar{K}_2^{\bar{X}} \rangle$ and $\bar{N}_2$ is isomorphic to $SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$. Moreover, N_2 contains $N_K(Y)'$ and $\overline{N_K(Y)'} \cong N_K(Y)'/Y$. Thus $N_2 = N_K(Y)' D_2$ and (4.23) holds.

(4.24) *There exists a subgroup N_3 such that $N_2/Y = N_3/Y \times N_3{}^t/Y$. In particular, N_3/Y is isomorphic to $(N_2/Y) \cap C(t) \cong N_L(A_2)'/F$.*

Proof. As in the proof of (4.23) we can write $N_K(Y)'/Y = N/Y \times N^t/Y$ with $N/Y \cong SL(2, 4)$. Set $B = D_2N$ and $Q = B \cap P$. Then $N_2/D_2 = B/D_2 \times B^t/D_2$ and $B/D_2 \cong SL(2, 4)$. Let $\overline{N(Y)} = N(Y)/Y$. Then $Z(\bar{P}) = \bar{V}_3$, $\bar{D}_2 = \bar{V}_3 \times \bar{V}_3{}^v$ by (4.17)(3), and $C_{N_2}(\bar{D}_2) = D_2$ since N_2/D_2 has no proper $\langle t \rangle$-invariant normal subgroups. It then follows that $\bar{D}_2 \geqslant Z(\bar{Q}) \geqslant \bar{V}_3$ and $Z(\overline{QQ^{vt}}) = Z(\bar{Q}) \cap Z(\bar{Q})^{vt} \neq 1$. Hence $Z(\bar{Q}) \neq \bar{V}_3$ and $Z(\bar{B}) = Z(\bar{Q}) \cap Z(\bar{Q})^v \neq 1$. Since B^t/D_2 acts nontrivially on $Z(\bar{B})$, [4, Lemma (4B)] implies $|Z(\bar{B})| = 16$. Thus $\bar{B}$ is a direct product of $Z(\bar{B})^t\bar{N}$ and $Z(\bar{B})$. Setting $N_3 = B'$, we get $N_2/Y = N_3/Y \times N_3{}^t/Y$. Finally, from (4.7) it follows that $\bar{N}_2 \cap C(t) = N_{N_2}(Y\langle t \rangle)/Y = N_L(A_2)' Y/Y \cong N_L(A_2)'/F$.

(4.25) $N(P) = N(D_1) \cap N(D_2)$.

Proof. By (4.17)(3) and (4.21) we have $N(P) \leqslant N(D_2) \leqslant N(Y)$. Moreover, it follows from (4.24) that P/Y has a unique elementary abelian subgroup of order 2^8 whose intersection with D_2/Y is equal to $Z(P/Y)$. Since D_1Y/Y is elementary abelian of order 2^8 and $D_1Y \cap D_2 = V_3$, this implies that $N(P) \leqslant N(D_1Y)$. Thus $N(P) \leqslant N(D_1)$ by (4.17)(5).

(4.26) $N(P\langle t \rangle) = N_H(S)P$. *In particular, if $H = LC_H(L)$ then $P\langle t \rangle$ is a Sylow 2-subgroup of G.*

Proof. Let $\overline{N(W)} = N(W)/W$. By (4.11), D_1 acts transitively on $\bar{A}_1\bar{t}$, so $[\bar{D}_1, \bar{t}] = \bar{A}_1$. Hence we have $(D_1\langle t \rangle)' = A_1W$. Similarly, (4.15)(1) and (4.17)(3) show that $(D_2\langle t \rangle)' = D_0$. So $(P\langle t \rangle)'$ contains $A_1WD_0 = SY$. If $(P\langle t \rangle)' = SY$, then $(RD_2)' \leqslant SY \cap D_2\langle t \rangle = D_0$, which contradicts (4.15)(3). Hence $(P\langle t \rangle)' > SY$. Moreover, $(P\langle t \rangle)' \leqslant R_3 \cap P = S_3$ and $S_3/SY \cong D_3/A_1V_2 \cong S'/F$ as $N_L(S)$-modules by (4.13)(4). Thus we conclude that $(P\langle t \rangle)' = S_3$. As S_3' is contained in $R_2 \cap V_3 = S'V_2$, (4.18)(1) yields that $C_{S_3}(S_3') = Y$. Therefore, V_3 is a characteristic subgroup of $P\langle t \rangle$ by the definition of V_3. Since P/V_3 is a direct product of D_1Y/V_3 and D_2/V_3 and so is elementary abelian, and since $C_P(V_3\langle t\rangle/V_3) = N_P(V_3\langle t\rangle) = S_3$ by (4.18)(3), it follows that $\mathscr{E}^*(P\langle t\rangle/V_3) = \{P/V_3, R_3/V_3\}$. Hence $N(P\langle t\rangle) = N_H(S)P$ by (4.20). The proof is complete.

5. The Case $K \cong PSL(4, 4)$ and $H \neq LC_H(L)$

In this section we assume the following hypothesis.

(5.1) *Hypothesis.* $K \cong PSL(4, 4)$ and $H \neq LC_H(L)$.
We will show

(5.2) $E(G) \cong PSL(5, 4)$.

Proof. By our hypothesis $H/C_H(L)$ is isomorphic to $\mathrm{Aut}(PSU(5, 2))$, so $H/C_H(L)$ has a subgroup $B/C_H(L)$ of order 2 which acts on L as a field automorphism and normalizes both S and $\langle e_2\rangle$. Let T be a Sylow 2-subgroup of B. Then $|T| = 4$ and T normalizes S and $\langle e_2\rangle$. In particular, T acts on K. If T is cyclic, then $t \in KC(K)$ since $\mathrm{Out}(PSL(4, 4))$ is elementary abelian, which contradicts (4.6)(3). Thus we can write $T = \langle t, b\rangle$ for some involution b, so that $H = \langle b\rangle LC_H(L)$ and b acts on L as a field automorphism. Now $C_K(t) = K_3 \cong PSU(4, 2)$ and $C_{K_3}(b) \cong Sp(4, 2)$, so $K \cap T = 1$ and KT is isomorphic to $\mathrm{Aut}(PSL(4, 4))$. Moreover, $|Sp(4, 2)|$ does not divide $|Sp(4, 4) \cap C(d)|$ where d is a transvection of $Sp(4, 4)$. Thus by (2.4) one of the following two cases occurs:

(i) b acts on K as a field automorphism and tb acts on K as a graph automorphism.

(ii) b acts on K as a graph automorphism and tb acts on K as a field automorphism.

Set $a = tb$ if in Case (i) and $a = b$ if in Case (ii).

Let D be an elementary abelian subgroup of ST of order 2^5. As the rank of $ST/\langle t\rangle$ is 4, $t \in D$. If $D \leqslant S\langle t\rangle$, then $D = F\langle t\rangle$ and $N_H(D)/C(D)$ is a symmetric group of degree 5. If $D \nleqslant S\langle t\rangle$ then $D \cap Sb \neq \varnothing$. By (3.10)(1) every involution of Sb is conjugate to b in $S\langle b\rangle$, so transforming D by an element of S if necessary, we may assume that $b \in D$. Moreover, (3.10)(2) implies that $C_S(b)$ has precisely two maximal elementary abelian subgroups B_1 and B_2, each of which is of order 8. Hence $D = B_1T$ or B_2T. Then $N_H(D) = N_L(D)\, TC_{O(H)}(D)$ and (3.11) shows that $N_H(D)/C(D)$ is isomorphic to a semidirect product of an elementary abelian group of order 8 and $SL(2, 2)$. On the other hand, ta acts on K as a field automorphism and $C_K(ta) \cong SL(4, 2)$. Thus $C_K(ta)\langle ta\rangle$ has a self-centralizing elementary abelian subgroup A of order 2^5 such that $C_K(ta)\langle ta\rangle \cap N(A)/A \cong SL(2, 2) \times SL(2, 2)$. Hence $C(ta) \cap N(A)/C(A)$ involves $SL(2, 2) \times SL(2, 2)$. Consequently, t and ta are not conjugate in G.

Let $X = C(a)$ and $\bar{X} = X/\langle a\rangle$. Then the above implies $C_{\bar{X}}(\bar{t}) = \overline{C_X(t)}$. Since $H = \langle a\rangle LC_H(L)$ and a induces a field automorphism of L, it follows that $C_{\bar{X}}(\bar{t}) = \overline{C_L(a)} \times \langle \bar{t}\rangle \times O(C_{\bar{X}}(\bar{t}))$ with $\overline{C_L(a)} \cong Sp(4, 2)$. Now $C_K(a) \cong Sp(4, 4)$ by the definition of a and t acts on $C_K(a)$ as a field automorphism. Moreover, as $C_K(t) \leqslant L$, $\overline{C_K(a)} \cap C(\bar{t}) = \overline{C_K(a) \cap C_X(t)} \leqslant \overline{C_L(a)}$. Therefore, applying [8, Lemma (1P)] to $\bar{X}$, we have $E(\bar{X}) \cong Sp(4, 4)$ and $C_{\bar{X}}(E(\bar{X})) = O(\bar{X})$. Since the Schur multiplier of $Sp(4, 4)$ is trivial, this yields that $E(X) \cong Sp(4, 4)$ and $C_X(E(X)) = \langle a\rangle\, O(X)$. Hence, $E(X)$ is a standard subgroup of G and $\langle a\rangle$ is a Sylow 2-subgroup of $C(E(X))$. Therefore, [7] shows that $E(G) \cong PSL(5, 4)$. The proof is complete.

6. The Case $K \cong PSL(4, 4)$ and $H = LC_H(L)$

In this section we argue under the following hypothesis.

(6.1) *Hypothesis.* $K \cong PSL(4, 4)$ and $H = LC_H(L)$.

By (4.6), $N_K(Y)$ has a subgroup L_2 such that $N_K(Y)' = (L_2 \times L_2{}^t)Y$ and $L_2 \cong SL(2, 4)$. In view of the proof of (4.24) we may assume that $L_2 \leqslant N_3$. Then $N_3 = L_2O_2(N_3)$ and $D_2 = O_2(N_3)\, O_2(N_3)^t$.

(6.2) Definition. Let L_2 be as above. Set $Q = L_2 \cap P$, $B = O_2(N_3)$, and $U = [B, e_2] \cap C(Q)$.

(6.3) *The following conditions hold.*

(1) $B = [B, e_2] \times Y$ *is elementary abelian of order* 2^{12}.

(2) $[B, e_2]$ *is a natural module for* $L_2 \cong SL(2, 4)$ *and* $\mathscr{E}^*(Q[B, e_2]) = \{QU, [B, e_2]\}$. *Moreover,* $L_2{}^t$ *centralizes* $[B, e_2]$.

(3) $\mathscr{E}^*(D_2) = \{B, B^t\}$.

(4) $D_1 = QUQ^tU^tV_2$.

Proof. By (4.24), $L_2{}^t$ centralizes B/Y. Let I be a complement of Q in $N_{L_2}(Q)$. Then I^t acts fixed-point-freely on Y, so $B = C_B(I^t) \times Y$. This implies that B is elementary abelian. Now, e_2 normalizes B since $(L_2D_2)' = N_3$. As e_2 acts fixed-point-freely on D_2/Y, we have $[B/Y, e_2] = B/Y$ and (1) holds. As $N_3/Y \cong N_L(A_2)'/F$ by (4.24), (2) holds.

If h is an element of $B^t - Y$, then $N_3 \rhd C_B(h)$ since N_3 centralizes B^t/Y. As $Y = Z(D_2)$ and N_3 acts irreducibly on B/Y, we must have $C_B(h) = Y$. Hence (3) follows from [20, Lemma (2.1)].

As $P/Y = QB/Y \times Q^tB^t/Y$, (2) and (4.21) imply $V_3 = UU^tY$. Then, since $V_3 \leqslant D_1Y$ by (4.17)(7), $U = [U, e_2] \leqslant [V_3, e_2] \leqslant D_1$. Comparing orders, we have that $D_1 \cap D_2 = UU^tV_2$. Furthermore, e_2 acts fixed-point-freely on P/D_1Y and so $Q \leqslant D_1Y$. Hence $QQ^tUU^tV_2 \leqslant D_1Y$. Since $V_2 = Z(P \bmod W)$ by (4.21) and D_1/W is elementary abelian, it follows that $QQ^tUU^tV_2/W$ is elementary abelian of order 2^{12}. Thus (4) is a consequence of (4.17)(5).

(6.4) Definition. Let $N_1 = E(N(D_1) \bmod D_1)$.

(6.5) $N_1 = \langle P, P^u\rangle$ *and* $N_1/D_1 \cong SL(3, 4)$.

Proof. Let $V = N(D_1) \cap C(W)$. Then P is a Sylow 2-subgroup of V by (4.26). Let $\overline{N(D_1)} = N(D_1)/D_1$. By (6.3), $\mathscr{E}^*(\bar{P}) = \{\bar{B}, \bar{B}^t\}$ and $\bar{B}$ is of order 16. Since $C_K(W)' = \langle MY, u\rangle$ and $\langle MY, u\rangle/M \cong SL(2, 4)$ by (4.17)(1), $\bar{V} \geqslant \langle \overline{MY}, \bar{u}\rangle \cong SL(2, 4)$.

Suppose $\bar{B}$ is strongly closed in $\bar{P}$ with respect to $\bar{V}$ and set $\bar{X} = \langle \bar{B}^{\bar{V}}\rangle$. Then a result of Goldschmidt [6] and the structure of $\bar{P}$ show that $\bar{X}/O(\bar{X}) = O_2(\bar{X}/O(\bar{X}))\, E(\bar{X}/O(\bar{X}))$ and each component of $\bar{X}/O(\bar{X})$ has abelian Sylow 2-subgroups. If $E(\bar{X}/O(\bar{X})) = 1$, then $\bar{B}O(\bar{V}) \lhd \bar{V}$. As $\bar{P} = \overline{BB^t}$, this implies that $\bar{V}$ is solvable of 2-length one, a contradiction. Hence $E(\bar{X}/O(\bar{X})) \neq 1$. Now, $E(\bar{V}/O(\bar{V}))$ is a product of $E(\bar{X}O(\bar{V})/O(\bar{V}))$ and $E(\bar{X}O(\bar{V})/O(\bar{V}))^t$ and $\bar{P}O(\bar{V})/O(\bar{V})$ is a Sylow 2-subgroup of $O_2(\bar{V}/O(\bar{V})) \times E(\bar{V}/O(\bar{V}))$, so that $|Z(\bar{P})| > 4$. But $Z(\bar{P}) = \bar{B} \cap \bar{B}^t = \bar{Y}$ has order 4. This contradiction implies that $\bar{B}$ is not strongly closed in $\bar{P}$ with respect to $\bar{V}$. Thus by [7, Lemma (1H)], $O^{2'}(\bar{V})\, O(\bar{V})/O(\bar{V}) \cong PSL(3, 4)$.

Set $\bar{A} = O^{2'}(\bar{V})$. Then $\bar{A}$ is perfect and $\bar{A}/O(\bar{A}) \cong PSL(3, 4)$. Since $H = LC_H(L)$, (4.18) yields that $C_{\bar{V}}(\bar{t}) = N(D_1\langle t\rangle) \cap C(W) = N_L(A_1)\, D_1O(H)$. As $N_L(A_1)$ contains $K_1 = \langle X_1, u\rangle \cong SU(3, 2)$, $\bar{A} \geqslant \bar{K}_1 \cong SU(3, 2)$. From (2.4) it follows that $\bar{t}$ acts on $\bar{A}/O(\bar{A})$ as a graph-field automorphism and $(\bar{A}/O(\bar{A})) \cap C(\bar{t}) = C_{\bar{A}}(\bar{t})\, O(\bar{A})/O(\bar{A}) \cong PSU(3, 2)$. Therefore, $\bar{K}_1 \cap O(\bar{A}) = Z(\bar{K}_1)$, and $\bar{K}_1$ stabilizes the series $O(\bar{A}) \cap C(\bar{t}) \geqslant Z(\bar{K}_1) \geqslant 1$, for $\bar{K}_1 \lhd C_{\bar{V}}(\bar{t})$ and $\bar{A} \lhd \bar{V}$. Hence $\bar{K}_1$ in fact centralizes $O(\bar{A}) \cap C(\bar{t})$. Now, $[\bar{Y}, \bar{t}] = Z(\bar{X}_1) \leqslant \bar{K}_1$. Applying Lemma (1J) of [7] to $\bar{A}\langle\bar{t}\rangle$, we have that $[O(\bar{A}), [\bar{Y}, \bar{t}]] = 1$. In particular, $C_{\bar{A}}(O(\bar{A})) \geqslant [\bar{Y}, \bar{t}]$ and so $\bar{A} = C_{\bar{A}}(O(\bar{A}))\, O(\bar{A})$. Since $\bar{A}$ is perfect, this implies $O(\bar{A}) = Z(\bar{A})$. In view of the Schur multiplier of $PSL(3, 4)$, we conclude that $\bar{A} \cong SL(3, 4)$.

By (4.17)(2), $V \lhd N(D_1)$ and as $P\langle t\rangle \in \mathrm{Syl}_2(N(D_1))$, we have $N_1/D_1 = \bar{A}$. Finally, $C_P(\bar{t}) = N_P(D_1\langle t\rangle) = SD_1$ and $S \cap S^u = A_1$, so $P \cap P^u = D_1$. Thus $N_1 = \langle P, P^u\rangle$.

(6.6) Definition. By (2.5) and (4.17)(1) there are two subgroups A_0 and M_0 of M which satisfy the following four conditions.

(1) $\mathscr{E}^*(M) = \{A_0, A_0^t\} \cup \{M_0^g, V_2^g \mid g \in N_K(M)\}$.

(2) $|A_0| = |M_0| = 2^6$, $M = A_0A_0{}^t$, $A_0 \cap A_0{}^t = W$, and $A_0 \cap M_0 = W$.

(3) $A_0 \lhd N_K(M)$.

(4) $M_0 \lhd N_K(M)'$ and M_0/W is a natural module for $N_K(M)'/M \cong SL(2, 4)$.

Note that $N_K(M)' = \langle MY, u\rangle$ and that (4) implies $|M_0 \cap V_2/W| = 4$ since $Z(MY/W) = V_2/W$. As L_2 normalizes either A_0 or $A_0{}^t$ by the definition of L_2, we may assume that

(5) $L_2{}^t$ normalizes A_0.

Then, both $A_0 \cap Y$ and $Y/A_0 \cap Y$ are natural modules for $L_2{}^t \cong SL(2, 4)$. Furthermore, t acts on the set $\{M_0{}^g \mid g \in N_K(M)\}$, which consists of odd number of elements. So t normalizes some $M_0{}^g$. Replace $M_0{}^g$ by M_0. Then we have

(6) $M_0{}^t = M_0$.

Finally, set $A = UA_0$.

(6.7) *The following conditions hold.*

(1) $A = QU(A_0 \cap Y)$ *is elementary abelian of order* 2^8, $D_1 = AA^t$, *and* $A \cap A^t = W$.

(2) $\langle N_1, L_2{}^t\rangle \leqslant N(A)$.

(3) $\mathscr{E}^*(P/Y) = \{D_1Y/Y,\ D_2/Y,\ AB^t/Y,\ A^tB/Y\}$ *and* $\mathscr{E}^*(D_1Y/W) = \{D_1/W, V_3/W\}$.

Proof. By (6.3)(4), we have $[D_1, e_2] \leqslant UU^tW$ and $M = QQ^tV_2$. Hence MY centralizes $[D_1, e_2]$ and so it follows that $\langle MY, u\rangle = \langle MY, Y^u\rangle$ centralizes $[D_1, e_2]$. In particular, $A = U \times A_0$ since $U \leqslant [D_1, e_2]$.

Let w be an involution in $L_2{}^t - N(Q^t)$. Then $M \cap M^w = A_0$ and as w centralizes Q, we have $A_0 = Q(A_0 \cap Y)$ and (1) holds. (6.6)(5) implies that $L_2{}^t$ normalizes A. By the above $\langle MY, u\rangle$ centralizes U and so it normalizes A. By (6.3)(2), $[B, e_2]$ normalizes QU, so B normalizes A. In addition, $D_1 \rhd A$ since D_1/W is abelian. Moreover, $N_3 = \langle L_2, B \cap D_1, Y\rangle$ and so $N_3{}^t$ normalizes A. Therefore, $P = D_1BB^t \rhd A$ and (2) holds. The first part of (3) is a consequence of (4.24), (6.3), and (1). Since $\mathscr{E}^*(MY/W) = \{M/W, Y/W\}$ and $D_1 = (D_1 \cap D_2)M$, the second part of (3) follows from (4.17).

(6.8) Definition. Let $L_1 = C_{N_1}(e_1)$ and $L_0 = (N_{L_1}(C_B(e_1)^t))'W$.

(6.9) *The following conditions hold.*

(1) $N_1 = L_1D_1$ *and* $L_1 \cap D_1 = W$.

(2) $O_2(L_0) = C_B(e_1)^t$ *has order* 2^6, $L_0/O_2(L_0) \cong SL(2, 4)$, *and* $C_P(e_1) = C_B(e_1)\, C_B(e_1)^t \in \mathrm{Syl}_2(L_0)$.

(3) L_0 *normalizes* B^t *and centralizes* $A \cap Y$. *Moreover, both* $(A \cap B)^t/W$ *and* $A/A \cap Y$ *are natural modules for* $L_0/O_2(L_0) \simeq SL(2, 4)$.

(4) L_1 *acts transitively on* $(A/W)^{\#}$ *and* $\langle L_1, L_2{}^t \rangle$ *acts transitively on* $A^{\#}$.

Proof. By (4.11), $C_{D_1}(e_1) = W$. Moreover, e_1 centralizes P/D_1Y and D_1Y/D_1, so $[P, e_1] \leqslant D_1$. Thus (6.5) implies (1). By (6.3)(3), e_1 normalizes B and so $C_P(e_1) = C_{D_2}(e_1) = C_B(e_1)\, C_B(e_1)^t$. As $C_P(e_1)$ is a Sylow 2-subgroup of L_1 and $L_1/W \simeq SL(3, 4)$, it follows that $\mathscr{E}^*(C_P(e_1)/W) = \{C_B(e_1)/W, C_B(e_1)^t/W\}$ and $N_{L_1}(C_B(e_1)^t)/W$ is a maximal 2-local subgroup of L_1/W. Thus (2) holds.

Let $T = [B, e_2]$. As $[T, e_1] \leqslant [P, e_1] \leqslant D_1$ and $C_T(e_1) \cap D_1 \leqslant T \cap W = 1$, $[T, e_1] = T \cap D_1 = U$. Then (6.3)(2) implies $C_{QU}(C_T(e_1)) = U$. Since $C_B(e_1) = C_T(e_1)\, C_Y(e_1)$ by (6.3)(1), it follows from (6.7)(1) that $C_A(C_B(e_1)) = A \cap B$, which is normalized by $L_0{}^t$. Since $A^tB = A^t(A \cap B)\, C_B(e_1)$, $L_0{}^t$ normalizes A^tB as well. We have $B = A^tB \cap C(A \cap B)$ by (6.3)(3), and so $L_0{}^t$ normalizes B. Now $A \cap B^t = A \cap Y$, on which L_0 acts. As $C_P(e_1)$ centralizes Y, (2) implies that L_0 centralizes $A \cap Y$. As $C_B(e_1) = C_T(e_1)\, C_Y(e_1)$, $(A/A \cap Y) \cap C(C_P(e_1)) = U(A \cap Y)/A \cap Y$ by (6.3)(2). Hence (2.3) shows that $A/A \cap Y$ is a natural module for $L_0/O_2(L_0) \simeq SL(2, 4)$. Similarly, $L_0/O_2(L_0)$ acts on $(A \cap B)^t/W$ and $C_P(e_1)$ centralizes $(A \cap Y)^t/W$. If $C_P(e_1)$ centralizes $(A \cap B)^t/W$, then $(A \cap B)^t \leqslant Z(P \bmod W)$ since $P = D_1C_P(e_1)$ and D_1/W is abelian. But this conflicts with (4.21). Thus $(A \cap B)^t/W$ is a natural module for $L_0/O_2(L_0) \simeq SL(2, 4)$ and (3) holds. In particular, L_0 acts transitively on $(A/A \cap Y)^{\#}$ and $L_0{}^t$ acts transitively on $(A \cap B/W)^{\#}$, so that the first part of (4) holds. As $L_2{}^t$ acts transitively on $(A \cap Y)^{\#}$, the second part of (4) holds.

(6.10) Definition. Let $G_0 = O^2(G)$.

(6.11) *P is a Sylow* 2-*subgroup of* G_0.

Proof. Let x be an involution of P. We argue that the 2-rank of $C(x)$ is greater than or equal to 8. Indeed, $N_1/D_1 \simeq SL(3, 4)$ has only one conjugacy class of involutions, so we may assume $x \in D_1Y$. Then $x \in D_1 \cup V_3$ by (6.7), and as $[V_3, Y] = 1$, we assume $x \in D_1 - A$. Since L_1 acts transitively on the nonidentity elements of $D_1/A \simeq A^t/W$ by (6.9)(4), we may assume $x \in AV_2$. Now $AV_2 = A_0V_2 \times U$ and $\mathscr{E}^*(A_0V_2) = \{A_0, V_2\}$. Thus $x \in A \cup V_2U$, and we conclude that $m(C(x)) \geqslant 8$. In particular, $t^G \cap P = \varnothing$ since $m(C(t)) = 5$. As $P\langle t \rangle$ is a Sylow 2-subgroup of G, (6.11) follows by the Thompson transfer lemma.

(6.12) $J_r(P) = D_2$ *and B is weakly closed in P with respect to* G_0.

Proof. Let E be an elementary abelian subgroup of P of order at least 2^{12}. We wish to show that $E = B$ or B^t. If $E \leqslant D_1Y$, then E lies in V_3 or D_1 by (6.7)(3). As V_3 is nonabelian of order 2^{12}, we have $E \leqslant D_1$. But then

$|D_1 : A_1| = 2^7$ implies $|E \cap A_1| \geqslant 2^5$, which is a contradiction. Thus $E \not\leqslant D_1 Y$. Suppose $B^t \neq E \leqslant AB^t$ and take $x \in E - B^t$. As $A \cap B^t = A \cap Y$, (6.9) shows that L_0 acts transitively on $(AB^t/B^t)^{\#}$, and so $x^g \in (A \cap B)B^t - B^t$ for some $g \in L_0$. Then $E^g \cap D_2 \leqslant C(x^g) \cap D_2 = B$ by (6.3)(3). Now $AB^t \cap B = (AB^t \cap D_2) \cap B = (A \cap B)Y$ has order 2^{10}, and $|AB^t: AB^t \cap D_2| = 4$ implies $|E^g \cap D_2| \geqslant 2^{10}$. Hence we have $E^g \cap D_2 = (A \cap B)Y$. But then $E^g \leqslant C_P(Y) = D_2$ by (4.21), a contradiction. Thus $E \leqslant AB^t$ implies $E = B^t$. By symmetry, if $E \leqslant A^t B$ then $E = B$. Therefore, (6.3)(3) and (6.7)(3) yield that $E = B$ or B^t as required. It then follows that $J_r(P) = BB^t = D_2$ since $C_P(B) = B$. As $N_{G_0}(P)$ acts on the set $\mathscr{E}^*(D_2) = \{B, B^t\}$ and $B \lhd P \in \mathrm{Syl}_2(G_0)$, B is normal in $N_{G_0}(P)$. Thus B is weakly closed in P with respect to G_0 by Burnside's lemma [9, 7.1.1].

(6.13) *Let x be an involution in P.*

(1) *If $x \in D_1$ then x is fused into V_2 by an element of N_1.*

(2) *If $x \in D_2$ then x is fused into $D_1 \cap D_2$ by an element of N_2.*

(3) *x is fused into V_2 by an element of G_0.*

Proof. Suppose $x \in D_1$. As in the proof of (6.11), x is L_1-conjugate to an element $y \in A \cup V_2 U$. If $y \in A$, then $y^{L_1} \cap V_2 \neq \varnothing$ by (6.9)(4). If $y \in V_2 U$, then (6.9)(3) shows that y is fused into $V_2 = (A \cap Y)(A^t \cap Y)$ by an element of $L_0{}^t$. Thus (1) holds. For the proof of (2), we may assume $x \in B$. Then, $x^g \in (A \cap B)Y$ for some $g \in L_2$ by (6.3)(2). Moreover, $(A \cap B)Y/A \cap B \cong Y/A \cap Y$ is a natural module for $L_2{}^t \cong SL(2, 4)$. Thus $x^{gh} \in (A \cap B)V_2$ for some $h \in L_2{}^t$ and (2) holds. Since every involution of P is conjugate to an element of $D_1 Y$ in N_1, (3) is a consequence of (1), (2), and (6.7)(3).

(6.14) *If $x \in W^{\#}$, then*

(1) $x^{G_0} \cap D_1 \cap D_2 = (A \cap B)^{\#} \cup (A^t \cap B^t)^{\#}$;

(2) $x^{G_0} \cap V_3 \cap B = (A \cap B)^{\#} \cup \{y^g \mid y \in (A^t \cap Y)^{\#}, g \in L_2{}^t\}$;

(3) $x^{G_0} \cap UU^t M_0 = (UW)^{\#} \cup (U^t W)^{\#}$.

Proof. Set $P_0 = QD_2$. As $N_2 \cap C(D_2/Y) = D_2$ and $Z(D_2) = Y$, it follows that $Z(P_0 P_0^{vt}) \leqslant Z(P_0) \leqslant C_Y(Q) = A \cap Y$. Also, $W^v \cap V_2 = 1$ since $W^v \cap V_2 \cap H = Z(S)^v \cap Z_2(S) = 1$. If P_2 is a Sylow 2-subgroup of N_2 different from P, then (4.23) shows that P_2 is conjugate by an element of P to P^v, $P_0 P_0^{vt}$, or $P_0{}^v P_0{}^t$. Therefore, we have $Z(P_2) \cap V_2 \leqslant (A \cap Y) \cup (A^t \cap Y)$.

Let $x \in W^{\#}$. By (6.12), D_2 is weakly closed in P with respect to G_0, so $N_{G_0}(D_2)$ controls the G_0-fusion of elements of Y. As $W = Z(P)$ and $N_2 \lhd N_{G_0}(D_2)$, the above then yields that $x^{G_0} \cap V_2 = (A \cap Y)^{\#} \cup (A^t \cap Y)^{\#}$.

Let z be an element of $D_1 \cap B$ not contained in $(A \cap B) \cup (A^t \cap Y)$. Since $D_1 \cap B = (A \cap B)(A^t \cap Y)$, (6.9)(3) implies that $L_0{}^t$ acts transitively on

$(D_1 \cap B/A^t \cap Y)^{\#}$ and so there is an element $g \in L_0{}^t$ such that $z^g \in V_2 - \{(A \cap B) \cup (A^t \cap Y)\}$. In particular, z is not conjugate to x in G_0. Thus $x^{G_0} \cap D_1 \cap B = (A \cap B)^{\#} \cup (A^t \cap Y)^{\#}$. Since $\mathscr{E}^*(D_1 \cap D_2) = \{D_1 \cap B, D_1 \cap B^t\}$ by (6.3)(3), (1) holds. As $V_3 \cap B = (A \cap B)Y$ and $V_3 \cap B/A \cap B \cong Y/A \cap Y$ is a natural module for $L_2{}^t \cong SL(2, 4)$, every element of $V_3 \cap B$ is fused into $(A \cap B)\, V_2$ by an element of $L_2{}^t$. Hence (2) follows from (1).

Let $y \in x^{G_0} \cap UU^tM_0$. By (4.17) and (6.3)(4) we have $[D_1/W, e_2] = UU^tW/W$ and $M = QQ^tV_2$. Hence $\langle MY, u\rangle = \langle MY, Y^u\rangle$ centralizes UU^tW. Moreover, M_0/W is a natural module for $\langle MY, u\rangle/M \cong SL(2, 4)$ and $M_0 \cap V_2 \neq W$ by (6.6). Thus $y^h \in UU^tV_2$ for some $h \in \langle MY, u\rangle$. Then (1) implies that y^h lies in $A \cap B$ or $(A \cap B)^t$. If $y^h \in A \cap B$, then $y^h \in UU^tM_0 \cap A \cap B = U(UU^tM_0 \cap A \cap Y)$. As $UU^tM_0 \cap Y \leqslant M_0$ and $M_0 \cap A = W$ by (6.6)(2), we have $y^h \in UW$. Similarly, if $y^h \in (A \cap B)^t$ then $y^h \in U^tW$. Since h centralizes UU^tW, (3) holds.

(6.15) *A is the unique elementary abelian subgroup of $AA_0{}^t$ of order 2^8 all of whose nonidentity elements are G_0-conjugate.*

Proof. Since $M = QQ^tV_2$ and $[Q^t, U] = 1$ by (6.3), $AA_0{}^t = U \times M$. Let E be an elementary abelian subgroup of $AA_0{}^t$ of order 2^8. Then by (6.6), E is equal to A, $UA_0{}^t$, $UM_0{}^g$, or $UV_2{}^g$ for some $g \in N_K(M)$. Let $x \in W^{\#}$. (6.14)(1) implies $x^{G_0} \cap V_2 \neq V_2{}^{\#}$, so we may assume $E \neq UV_2{}^g$. Similarly, (6.14)(3) implies $x^{G_0} \cap M_0 \neq M_0{}^{\#}$ and we may assume $E \neq UM_0{}^g$. Finally, $U(A^t \cap Y)$ is a subgroup of $UA_0{}^t \cap D_1 \cap D_2$ not contained in $A \cup A^t$ so that $x^{G_0} \cap UA_0{}^t \neq (UA_0{}^t)^{\#}$ by (6.14)(1).

(6.16) *A is weakly closed in P with respect to G_0.*

Proof. Since $N_1 \cap C(D_1/A^t) = D_1$ and since L_1 acts irreducibly on D_1/A, it follows that $C_P(A) = A$ and so $C_{G_0}(A) = A \times O(C_{G_0}(A))$. In particular, D_2 does not contain any G_0-conjugate of A.

Suppose $A^g \leqslant AB^t$ for some $g \in G_0$. As $C_P(B) = B$, we have $A \cap Y = A \cap B^t = Z(AB^t) \leqslant C(A^g)$, so $A \cap Y \leqslant A^g$. As $L_2D_2 = \langle QD_2, Q^v\rangle$, it follows that $A \cap Y \cap A^v \leqslant C_Y(L_2) = 1$, and hence $Y = (A \cap Y) \times (A \cap Y)^v$. If $A^g \cap Y \neq A \cap Y$, then $A^g \cap A^v \cap Y$ has an element $z \neq 1$. Since $L_2D_2 \neq L_2D_2 \cap C(z) \geqslant Q^vD_2$, $QD_2 \cap C(z) = D_2$. But this is impossible since $z \in A^g \leqslant QD_2$ and $A^g \nleqslant D_2$. Thus $A^g \cap Y = A \cap Y$. As $L_2{}^t$ normalizes Y, (6.14)(2) shows that $A^g \cap V_3 \cap B \leqslant (A \cap B) \cup Y$. Hence $A^g \cap Y = A \cap Y$ implies $A^g \cap V_3 \cap B \leqslant A \cap B$. If $A^g \cap D_1Y \leqslant V_3 \cap B$, then $A^g \cap D_1Y = A \cap B$ since $AB^t \cap D_1Y$ is a subgroup of AB^t of index 4 and $|A^g \cap D_1Y| \geqslant |A \cap B|$. But then $A^g \leqslant AB^t \cap C(A \cap B) = AY$, a contradiction. Thus $A^g \cap D_1Y \nleqslant V_3 \cap B$. Transforming each side of the equation of (6.14)(2) by t, we have $A^g \cap V_3 \cap B^t \leqslant (A \cap B)^t \cup Y$. Hence $A^g \cap Y = A \cap Y$ implies $A^g \cap V_3 \cap B^t \leqslant (A \cap B)^t \cup (A \cap Y)$. If $A^g \cap D_1Y \leqslant V_3 \cap B^t$, then

$A^g \cap D_1Y \leqslant (A \cap B)^t \cup (A \cap Y)$. But $A^g \cap D_1Y$ is of order at least 2^6 and contains $A \cap Y$, a contradiction. Thus $A^g \cap D_1Y \nleqslant V_3 \cap B^t$. It then follows that $A^g \cap D_1Y \nleqslant V_3$ since $\mathscr{E}^*(V_3) = \{V_3 \cap B, V_3 \cap B^t\}$. Therefore, $A^g \cap D_1Y \leqslant D_1$ by (6.7)(3). If h is an element of L_0, then $A^{gh} \leqslant AB^t$ and the above argument shows that $A^g \cap (D_1Y)^{h^{-1}} \leqslant D_1^{h^{-1}} = D_1$. By (6.9), $D_1B^t/D_1 \simeq C_B(e_1)^t/W$ is a natural module for $L_0/O_2(L_0) \simeq SL(2, 4)$. In particular, every element of D_1B^t is fused into D_1Y by an element of L_0. Therefore, $A^g \leqslant D_1$ since we are assuming that $A^g \leqslant AB^t$.

Suppose $A^g \leqslant P$ for some $g \in G_0$. We have shown that $A^g \leqslant AB^t$ implies $A^g \leqslant D_1$. By symmetry, if $A^g \leqslant A^tB$ then $A^g \leqslant D_1$. Thus, in view of (6.7)(3), we get that $A^g \leqslant D_1$. Since UU^tM_0 is a subgroup of D_1 of index 16, $| A^g \cap UU^tM_0 | \geqslant 16$. Furthermore, (6.14)(3) implies that $A^g \cap UU^tM_0$ is a subgroup contained in $UW \cup U^tW$. Hence $A^g \cap UU^tM_0$ is equal to UW or U^tW. Now $C_{D_1}(U) = AA_0{}^t$, so that $A^g = A$ or $A^g = A^t$ by (6.15). As $A \lhd P \in \mathrm{Syl}_2(G_0)$, we conclude that $A^g = A$. The proof is complete.

(6.17) *A is strongly closed in P with respect to $C_{G_0}(x)$ for $x \in W^\#$.*

Proof. Set $H_1 = C_{G_0}(x)$, $x \in W^\#$. We have $N_{G_0}(D_2) = N_{G_0}(P)\, N_2$ by the Frattini argument. As $N_{L_2}(Q)$ is a subgroup of $N_{N_2}(P)$ and acts transitively on $W^\#$, it follows that $N_{G_0}(D_2) = N_{H_1}(P)\, N_2$. Moreover, P is normal in $H_1 \cap N_2$. Thus $N_{H_1}(D_2) \leqslant N(P) \leqslant N(A)$.

Suppose that an element d of A is H_1-conjugate to an element e of $P - A$. Since $N_1 \leqslant N_{H_1}(A)$, we may assume that $d \in A \cap Y$. Also, N_1/D_1 has only one conjugacy class of involutions and we may assume $e \in D_1Y$. Then $e \in D_1 \cup (V_3 \cap B) \cup (V_3 \cap B^t)$ by (6.3)(3) and (6.7)(3) and hence $e \in D_1 \cup Y$ by (6.14)(2). In view of (6.13)(1) we may assume that $e \in Y$. Then d and e are conjugate in $N_{H_1}(D_2)$, for D_2 is weakly closed in P with respect to H_1. This is a contradiction.

(6.18) *The following conditions hold.*

(1) $N_{G_0}(W) = N_{G_0}(D_1)\, O(N_{G_0}(W))$.

(2) $C_{G_0}(x) \rhd WO(C_{G_0}(x))$ *and* $C_{G_0}(x)$ *is 2-constrained for* $x \in W^\#$.

Proof. Set $H_0 = C_{G_0}(W)$. Then by (6.17), A is strongly closed in P with respect to H_0. Let $\bar{H}_0 = H_0/O(H_0 \bmod W)$ and $X = \langle A^{H_0}\rangle$. A result of Goldschmidt [6] shows that $\bar{X} = O_2(\bar{X})\, E(\bar{X})$ and $\Omega_1(\bar{X} \cap \bar{P}) = \bar{A}$. Since L_1 is a subgroup of H_0 and acts transitively on $\bar{A}^\#$, $\bar{X} = \bar{A}$ or $\bar{X}$ is a simple group of type I or II in the sense of [6]. If $\bar{X}$ is simple, then L_1 centralizes $\bar{X}$, a contradiction. Thus $\bar{A} = \bar{X}$ and so $\bar{D}_1 = \bar{A}\bar{A}^t \lhd \bar{H}_0$. This implies that $H_0 = N_{H_0}(D_1)\, O(H_0)$. Now, $N_{G_0}(W) = H_0N_{G_0}(P)$ by the Frattini argument, $N(P) \leqslant N(D_1)$, and $N(D_1) \leqslant N(W)$ by (4.17)(2). Thus (1) holds.

For the proof of (2), set $H_1 = C_{G_0}(x)$, $x \in W^\#$. (6.17) implies that both A and A^t are strongly closed in P with respect to H_1. Hence $W = A \cap A^t$ is also strongly closed in P with respect to H_1. This time, let $\bar{H}_1 = H_1/O(H_1)$ and $X = \langle W^{H_1} \rangle$. Then by [6], $\bar{X} = O_2(\bar{X}) E(\bar{X})$ and $\Omega_1(X \cap P) = W$. If $W \neq X \cap P$, then $W < X \cap Z(P \bmod W)$. But $Z(P \bmod W) = V_2$ is elementary abelian so that $W < \Omega_1(X \cap P)$, a contradiction. Thus $W \in \mathrm{Syl}_2(X)$. It then follows that $\bar{X} = \bar{W}$ and $H_1 = N_{H_1}(W)\, O(H_1)$. Finally, $N_{G_0}(D_1)$ is 2-constrained and hence H_1 is 2-constrained as well.

(6.19) *QB/B is strongly closed in P/B with respect to $N_{G_0}(B)/B$.*

Proof. Let $N = N_{G_0}(B)$ and $\bar{N} = N/B$. Then $\bar{P} = \bar{Q} \times \bar{Q}^t \bar{D}_2$ and $\mathscr{E}^*(\bar{P}) = \{\bar{D}_1, \bar{Q}\bar{D}_2\}$. Hence $\bar{D}_1$ and $\bar{Q}\bar{D}_2$ are weakly closed in $\bar{P}$ with respect to $\bar{N}$ by Burnside's lemma. Take an element $\bar{d} \in \bar{Q}$ and a $\bar{N}$-conjugate $\bar{e} \in \bar{P}$ of $\bar{d}$. If $\bar{e} \in \bar{D}_1$, then $\bar{d}$ is conjugate to $\bar{e}$ by an element of $N_{\bar{N}}(\bar{D}_1)$. Since $A \lhd N(D_1B)$ by (6.16) and $QB = AB$, it follows that $\bar{e} \in \bar{Q}$. Likewise, $A \lhd N(QD_2)$ and so if $\bar{e} \in \bar{Q}\bar{D}_2$ then $\bar{e} \in \bar{Q}$. Thus $\bar{Q}$ is strongly closed in $\bar{P}$ with respect to $\bar{N}$.

(6.20) *If $x \in (M_0 \cap V_2) - W$, then $M_0D_2 \in \mathrm{Syl}_2(C_{G_0}(x))$ and B is strongly closed in M_0D_2 with respect to $C_{G_0}(x)$.*

Proof. Let $H_2 = C_{G_0}(x)$ with $x \in (M_0 \cap V_2) - W$ and $V = M_0D_2$. If Q_2 is a Sylow 2-subgroup of H_2 containing V, then $D_2 \lhd Q_2$ by (6.12) and so $Q_2 \leqslant N_2$. Since M_0Y is a Sylow 2-subgroup of $C_K(x)$ and since $N_2 = N_K(Y)'D_2$, we conclude that V is a Sylow 2-subgroup of H_2. As $C_A(x) = A \cap B$, we have $AB^t \cap V \leqslant D_2$. Since t normalizes M_0 by (6.6)(6), it then follows from (6.7)(3) that $\mathscr{E}^*(V/Y) = \{D_1Y \cap V/Y, D_2/Y\}$ and each member of $\mathscr{E}^*(V)$ is contained in either D_1 or D_2.

We now proceed as in Lemma (8N) of [7]. Suppose an element $d \in V - B$ is fused into B by an element of H_2. Choose $g \in H_2$ such that

(1) $d^g \in B$ and

(2) $|B \cap B^{g^{-1}}|$ is maximal subject to (1).

Set $T_1 = B \cap B^{g^{-1}}$, $T_2 = N_B(\langle T_1, d\rangle)$, $T_3 = C_B(d)$, and $T_0 = [B, d]$. Then by (6.12) and [7, Lemma (1G)] we have

(3) $T_0T_1 \leqslant T_3 \leqslant T_2$, $T_2^g \cap B = T_1^g \neq T_2^g$, $|B/T_3| = |T_0| \leqslant |T_2/T_1|$, and $|T_0 \cap T_1| = |T_2/T_3|$.

Furthermore, we may assume that

(4) $N_V(\langle T_1, d\rangle)^g \leqslant V$.

By (4), T_2^g is an elementary abelian subgroup of V, so it is contained in D_1 or D_2. Hence we have $|T_2^g/T_2^g \cap B| \leqslant 16$, for $|D_2 : B| = |D_1 \cap V : D_1 \cap B| =$

16. So $|T_2/T_1| \leqslant 16$ by (3). If $d \notin D_2$, then $C_V(d) \leqslant D_1$ and $T_3 \leqslant D_1 \cap B$. As $|B/T_3| \leqslant |T_2/T_1|$, it follows that $T_3 = D_1 \cap B$. But then $d \in C_P(T_3) \leqslant C_P(V_2) = D_2$ by (4.21), a contradiction. Thus (6.3)(3) implies that $d \in B^t$ and $T_3 = Y$, and hence $|T_2/T_1| = 16$. Also, $\langle T_1, d\rangle \leqslant B^t$ and (4) shows that $B^{tg} \leqslant V$. Therefore, $g \in N(B^t)$ by (6.12). If $T_2{}^g \leqslant D_1$, then $Y^g = D_1 \cap B^t$ since $Y \leqslant T_2 \cap B^t$. However, $D_1 \cap B^t \lhd V \in \mathrm{Syl}_2(H_2)$, D_2 is weakly closed in V with respect to H_2, and $Z(D_2) = Y$, so that [7, Lemma (1F)] yields $Y = D_1 \cap B^t$, which is a contradiction. Thus $T_2{}^g \leqslant D_2$. As $T_2{}^g \nleqslant B$, it follows that $T_2{}^g \leqslant B^t$. Hence $T_2 \leqslant B^t$ and we have $T_2 = B \cap B^t = Y$. Then (3) implies that $Y = T_0 \times T_1$ and $|T_1| = 16$. Since $g \in N(B^t)$ and $T_1 = B \cap B^{g^{-1}} \leqslant Y$, it follows that $T_1{}^g \leqslant Y$ and $T_1 = Y \cap Y^{g^{-1}}$. In particular, $B^t = YY^{g^{-1}}$.

Let e be an element of $Y^{g^{-1}} - Y$ and choose $h \in H_2$ such that $e^h \in B$ and $|B \cap B^{h^{-1}}|$ is maximal subject to $e^h \in B$. Notice that $e \in V - B$ and $e^g \in B$. We apply the above argument with e and h in the role of d and g. Then $|T_1| = 16$ implies $|B \cap B^{h^{-1}}| = 16$ and $T_0 \cap T_1 = 1$ implies $[B, e] \cap B^{h^{-1}} = 1$. Since $e^g \in B$, we may choose $h = g$. Then $[B, e] \cap T_1 = 1$. As $B^t = YY^{g^{-1}}$, we get that $[B, a] \cap T_1 = 1$ for all $a \in B^t - Y$. If $a \in U^t$, then $1 \neq [U, a] \leqslant [B, a] \cap W$ by (6.3). Moreover, L_2 centralizes U^t and $A^t \cap Y$ is a natural module for $L_2 \cong SL(2, 4)$, so $A^t \cap Y \leqslant [B, a] \leqslant Y$. Since $[B, a] \cap T_1 = 1$, it follows that $[B, a] = A^t \cap Y$. Now $L_2{}^t$ acts transitively on $(B^t/Y)^{\#}$ so that

$$\{y \mid y \in [B, a], a \in B^t - Y\} = \{z^h \mid z \in A^t \cap Y, h \in L_2{}^t\}.$$

If $w \in W^{\#}$, then $w^{G_0} \cap Y = \{z^h \mid z \in (A^t \cap Y)^{\#}, h \in L_2{}^t\}$ by (6.14)(2) since $L_2{}^t$ acts transitively on $(A \cap Y)^{\#}$. Thus we have that $w^{G_0} \cap T_1 = \varnothing$.

Set $X = \langle D_2, D_2^{g^{-1}}\rangle$. Then $X \rhd B^t$ and $X \leqslant C_{G_0}(T_1) \leqslant H_2$ since $g \in N(B^t)$ and $x \in T_1 = Y \cap Y^{g^{-1}}$. Suppose $D_2 \notin \mathrm{Syl}_2(X)$ and let P_1 be a Sylow 2-subgroup of X containing D_2. Then $D_2 \lhd P_1$, so there is an element $k \in N_{H_2}(D_2)$ such that $D_2 < P_1{}^k \leqslant C_V(T_1{}^k)$. Then $C_{M_0}(T_1{}^k)$ is not contained in D_2, so $C_Y(C_{M_0}(T_1{}^k)) \leqslant V_2$. As $C_V(D_2) = Y$, it follows that $T_1{}^k \leqslant Z(C_V(T_1{}^k)) \leqslant V_2$. In particular, $(A \cap V_2) \cap T_1{}^k \neq 1$. However, this is impossible since $w^{G_0} \cap T_1 = \varnothing$ for $w \in W^{\#}$. Thus $D_2 \in \mathrm{Syl}_2(X)$. Then $D_2^{g^{-1}} = D_2{}^h$ for some $h \in X$ and so $Y \ntrianglelefteq X$ since $Y^g \neq Y = Z(D_2)$. Now (6.3)(3) and [7, Lemma (1H)] show that X/B^t has a strongly embedded subgroup.

Let $N = N_{G_0}(B^t)$ and $\bar{N} = N/B^t$. By (6.19), $\bar{Q}^t$ is strongly closed in $\bar{P}$ with respect to $\bar{N}$. Let $\bar{N}_0$ denote the normal closure of $\bar{Q}^t$ in $\bar{N}$. Then $\bar{Q}^t = \Omega_1(\bar{N}_0 \cap \bar{P})$ by [6]. If $\bar{Q}^t \notin \mathrm{Syl}_2(\bar{N}_0)$, then $\bar{Q}^t < \bar{N}_0 \cap Z(\bar{P} \bmod \bar{Q}^t)$. But this is a contradiction, for $Z(\bar{P} \bmod \bar{Q}^t)$ is elementary abelian. Thus $\bar{Q}^t \in \mathrm{Syl}_2(\bar{N}_0)$. Also, $\bar{N}_0/O(\bar{N}_0)$ is simple, for $\bar{N}_0$ contains $\bar{L}_2{}^t$. Set $\bar{X}_0 = C_{\bar{N}}(\bar{N}_0/O(\bar{N}_0))^{(\infty)}$. Then $\bar{L}_2 \leqslant \bar{X}_0$ since $\overline{Q \in \mathrm{Syl}_2(L_2)}$ and $\bar{Q} \cap \bar{Q}^t = 1$. Likewise $L_0 \leqslant N$ and $\bar{L}_0 \cong SL(2, 4)$ with $\overline{C_P(e_1)} = \overline{C_B(e_1)}$ a Sylow 2-subgroup by (6.9), and as $\overline{C_P(e_1)} \cap \bar{Q}^t = 1$, we have $\bar{L}_0 \leqslant \bar{X}_0$. Now $\bar{P} = \bar{Q}^t \times \overline{QB}$, $\mathscr{E}^*(\overline{QB}) = \{\overline{QU}, \bar{B}\}$, and $\langle \overline{Q}, \overline{C_P(e_1)} \rangle = \overline{QB}$. Thus $\overline{QB} = \overline{X}_0 \cap \overline{P} \in \mathrm{Syl}_2(\overline{X}_0)$. Moreover, L_2 acts transitively on $\overline{B}^{\#}$ and L_0 acts transitively on $(\overline{QU})^{\#}$ by (6.9), so $\overline{X}_0$ has only one conjugacy class of involutions and hence neither $\overline{QU}$ nor $\overline{B}$ is strongly closed in $\overline{QB}$ with respect to $\overline{X}_0$. Thus [7, Lemma (1H)] shows that $\overline{X}_0/O(\overline{X}_0) \simeq PSL(3, 4)$. Hence, $\langle D_2, D_2{}^y \rangle / B^t$ does not have a strongly embedded subgroup for any $y \in N$. However, as shown above X/B^t has a strongly embedded subgroup. This contradiction completes the proof.

(6.21) *The following conditions hold.*

(1) $N_{G_0}(Y) = N_{G_0}(D_2)\, O(N_{G_0}(Y))$.

(2) $C_{G_0}(x) \trianglerighteq YO(C_{G_0}(x))$ *and* $C_{G_0}(x)$ *is 2-constrained for* $x \in (M_0 \cap V_2) - W$.

Proof. Set $H_0 = C_{G_0}(Y)$. Then $D_2 \in \mathrm{Syl}_2(H_0)$ by (4.21) and B is strongly closed in D_2 with respect to H_0 by (6.20). Let $\overline{H}_0 = H_0/O(H_0 \bmod Y)$. Then $\overline{H}_0 \cap C(t)$ is solvable since $(C(Y)/Y) \cap C(t) = C(Y) \cap N(Y\langle t \rangle)/Y$ is solvable by (4.7). Set $X = \langle B^{H_0} \rangle$. Then [6] shows that $\overline{X} = O_2(\overline{X})\, E(\overline{X})$ and $\overline{B} \in \mathrm{Syl}_2(\overline{X})$ since $\overline{D}_2$ is elementary abelian. As $\overline{B} \cap \overline{B}^t = 1$, it follows that $E(\overline{X}) \cap E(\overline{X})^t = 1$ and $E(\overline{X})\, E(\overline{X})^t \cap C(t)$ is isomorphic to $E(\overline{X})$. Since $\overline{H}_0 \cap C(t)$ is solvable, we conclude that $\overline{B} = \overline{X}$. Hence $\overline{D}_2 = \overline{BB^t} \lhd \overline{H}_0$. Moreover, $O(H_0 \bmod Y) = YO(N_{G_0}(Y))$ and $N_{G_0}(Y) = N_{G_0}(D_2)\, H_0$ by the Frattini argument. Thus (1) holds.

For the proof of (2), set $H_2 = C_{G_0}(x)$, $x \in (M_0 \cap V_2) - W$. Since t normalizes M_0, (6.20) implies that both B and B^t are strongly closed in $M_0 D_2$ with respect to H_2. Hence $Y = B \cap B^t$ is also strongly closed in $M_0 D_2$ with respect to H_2. Let $\overline{H}_2 = H_2/O(H_2)$ and $X = \langle Y^{H_2} \rangle$. By [6], $\overline{X} = O_2(\overline{X})\, E(\overline{X})$ and $\Omega_1(X \cap M_0 D_2) = Y$. Since x is a noncentral involution of K, Y is normal in $C_K(x)$ and $C_1 = C_K(x)'$ is perfect by (2.6). Now, C_1 permutes the components of $E(\overline{X})$ and the number of components of $E(\overline{X})$ is at most 4, so C_1 normalizes each component of $E(\overline{X})$. Let $\overline{X}_1$ be a component of $E(\overline{X})$. If $\overline{X}_1$ is of type II in the sense of [6], then $|\overline{X}_1 \cap \overline{Y}| \leqslant 8$ and C_1 centralizes $\overline{X}_1 \cap \overline{Y}$. If $\overline{X}_1$ is of type I, then C_1 centralizes $\overline{X}_1 \cap \overline{Y}$ as well, for C_1 is perfect. Since $\bar{x} \in O_2(\overline{X})$ and since $|C_{\overline{Y}}(C_1)| = 4$ by (2.6), we conclude that $E(\overline{X}) = 1$. Thus $H_2 = N_{H_2}(Y)\, O(H_2)$. Finally, $N_{G_0}(D_2)$ is 2-constrained, so H_2 is also 2-constrained and (2) holds.

(6.22) *If* $O(G) = 1$, *then* $E(G) \simeq PSL(5, 4)$.

Proof. The centralizer of every involution of G_0 is 2-constrained by (6.13), (6.18), and (6.21). Since $O(G_0) = 1$, a result of Gorenstein and Walter [10] implies that every 2-local subgroup of G_0 is core-free. Let $z \in W^{\#}$ and $H_1 = C_{G_0}(z)$. Then $H_1 \trianglerighteq D_1$ by (6.18). Suppose $A^g \leqslant H_1$ for some $g \in G_0$. Then $A^g \leqslant N_1$ since $N_{G_0}(D_1)/N_1$ has odd order. Choose an element $h \in N_1$ so that $A^{gh} \leqslant P$. Then $A^{gh} = A$ by (6.16) and so $A^g = A$. Since $N_{G_0}(A)$ acts transitively on $A^{\#}$, it follows that A is weakly closed in $C_{G_0}(a)$ with respect to G_0 for each $a \in A^{\#}$. Now, $E(G)$ is simple (see [20, Lemma (2.9)]), so a result of Timmesfeld [19] shows that $E(G) \cong PSL(5, 4)$.

Remark. The condition that $O(G) = 1$ in the above lemma will be removed in Section 8.

7. The Case $K \cong PSU(4, 2) \times PSU(4, 2)$

We retain the notation of Section 4. The purpose of this section is to prove that $E(G)$ is isomorphic to $PSU(5, 2) \times PSU(5, 2)$ under the following hypothesis.

(7.1) *Hypothesis.* $K \cong PSU(4, 2) \times PSU(4, 2)$.

(7.2) $N_2 = N_3' \times N_3'^t$ *and* N_3' *is isomorphic to* $N_2 \cap H = N_L(A_2)'$.

Proof. Take a component K_0 of K and set $E = N_{K_0}(Y)$ and $I = O_2(E)$. Then $N_K(Y) = E \times E^t$, $Y = I \times I^t$, and E is isomorphic to $N_K(Y) \cap H = K_2F$ by (4.6). So $N_K(Y)/Y = EY/Y \times E^tY/Y$. In view of the proof of (4.24) we may assume that $E \leqslant N_3$. Then $N_3 = (D_2 \cap N_3)E$. A Sylow 5-subgroup of K_2 acts fixed-point-freely on F, so $[I^t, X] = I^t$ for a Sylow 5-subgroup X of E^t. Since E^t centralizes N_3/Y by (4.24), we have $[N_3 \cap P, X] = [Y, X] = I^t$ and $N_3 \cap P$ is a direct product of I^t and $N_3 \cap P \cap C(X)$. Since $N_3 \cap P$ is a Sylow 2-subgroup of N_3 and I^t is contained in $Z(N_3)$, we can write $N_3 = I^t \times N$ for some subgroup N by Gaschütz's theorem. Then $E = E' \leqslant N_3' \leqslant N$ and $N \cap Y = I$. Thus $N_3 \cap N_3{}^t = Y$ implies that $N_3' \cap N_3'^t \leqslant I \cap I^t = 1$. As $D_2' = Y$, (7.2) now follows from (4.24).

(7.3) Definition. Let $Q = N_3' \cap P$, $G_1 = \langle Q, Q^u, Q^v \rangle$, and $G_0 = \langle P, P^u, P^v \rangle$.

(7.4) $G_0 = G_1 \times G_1{}^t$, $G_1 \cong PSU(5, 2)$, *and Q is a Sylow 2-subgroup of* G_1. *Furthermore,* $G_0 \cap H = L$ *and* $C(G_0)$ *has odd order.*

Proof. Set $B_1 = J_e(Q \bmod Z(Q))$ and $B_2 = O_2(N_3')$. Then (7.2) shows that

$$B_2 = Q \cap Q^v, \qquad D_2 = B_2 \times B_2{}^t, \qquad \text{and} \qquad N_3' = \langle Q, Q^v \rangle.$$

Moreover, $P = Q \times Q^t$ and Q is isomorphic to $C_P(t) = S$, so $W = Z(P) = Z(Q) \times Z(Q)^t$ and $B_1B_1{}^t/W$ is the unique elementary abelian subgroup of P/W of order 2^{12}. Thus

$$D_1 = B_1 \times B_1{}^t.$$

Since $N_L(A_1)$ normalizes D_1 and centralizes W, $N_L(A_1)$ normalizes $Z(D_1 \bmod Z(Q)) = B_1W$ as well. Now, e_1e_2 is an element of $N_{K_2}(S)$ and so normalizes Q. Consider the map defined by $x \mapsto xx^t$ for $x \in B_1$. This is a $\langle e_1e_2\rangle$-isomorphism of B_1 onto $D_1 \cap H = A_1$. As $[A_1, e_1e_2] = A_1$, we have $[B_1, e_1e_2] = B_1$. Since e_1e_2 centralizes u and W, it follows that u normalizes $[B_1W, e_1e_2] = B_1$. Moreover, $(P/D_1) \cap C(t) = SD_1/D_1$, so $(P \cap P^u/D_1) \cap C(t) = 1$. This implies that $P \cap P^u = D_1$ and hence

$$B_1 = Q \cap Q^u.$$

Define an isomorphism θ of N_3' onto $N_L(A_2)'$ by $x \mapsto xx^t$. This isomorphism maps B_1 onto A_1 and $B_1 \cap B_1{}^v$ onto $A_1 \cap A_1{}^v$. Hence $|B_1 \cap B_1{}^v| = 4$. Since $D_i \cap H = A_i$, $i = 1, 2$, and since $A_2 \cap (A_1 \cap A_1{}^v)^u = 1$, we have that $D_2 \cap (D_1 \cap D_1{}^v)^u = 1$. In particular, $B_2 \cap (B_1 \cap B_1{}^v)^u = 1$. Thus comparing orders we get

$$Q = B_2(B_1 \cap B_1{}^v)^u.$$

Since N_3' is generated by Q and Q^v, we have

$$N_3' = B_2\langle (B_1 \cap B_1{}^v)^u, (B_1 \cap B_1{}^v)^{uv}\rangle.$$

Let $X = \langle B_1, (B_1 \cap B_1{}^v)^{uv}\rangle$ and $\tilde{N}_3' = N_3'/Z(B_2)$. Then $N_3' = B_2X$. Since the isomorphism θ maps B_i onto A_i, $i = 1, 2$, and $A_1 \cap A_2 \nleqslant Z(A_2)$, it follows that $1 \neq \tilde{B}_1 \cap \tilde{B}_2 \leqslant \tilde{X} \cap \tilde{B}_2$. Also, $\tilde{X} \cap \tilde{B}_2$ is normal in $\tilde{N}_3'$ since $\tilde{B}_2$ is abelian. As $N_L(A_2)'$ acts irreducibly on $A_2/Z(A_2)$, N_3' acts irreducibly on $\tilde{B}_2$. Hence $\tilde{N}_3' = \tilde{X}$ and $N_3' = Z(B_2)X \rhd X \cap Z(B_2) \neq 1$. Moreover, N_3' acts irreducibly on $Z(B_2)$. Therefore,

$$N_3' = \langle B_1, (B_1 \cap B_1{}^v)^{uv}\rangle.$$

Now we have

$$\begin{aligned}
[B_1{}^u, B_1{}^t] &= [B_1, B_1{}^t] = 1,\\
[B_1{}^u, (B_1 \cap B_1{}^v)^{uvt}] &\leqslant [B_1{}^u, B_1^{uvt}] = [B_1, B_1^{vt}]\\
&\leqslant [N_3', N_3'^t] = 1,\\
[(B_1 \cap B_1{}^v)^{uvu}, B_1{}^t] &\leqslant [B_1^{uvu}, B_1{}^t] = [B_1^{vu}, B_1{}^t]\\
&= [B_1^{vt}, B_1]^{ut} = 1,\\
[(B_1 \cap B_1{}^v)^{uvu}, (B_1 \cap B_1{}^v)^{uvt}] &\leqslant [B_1^{vuvu}, B_1^{vuvt}]\\
&= [B_1^{vuvu}, B_1^{uvuvt}]\\
&= [B_1, B_1{}^t]^{(uv)^2} = 1,
\end{aligned}$$

where we use $(uv)^2 = (vu)^2$. Hence $[N_3'^u, N_3'^t] = 1$. As $N_3' = \langle Q, Q^v\rangle$, this implies that $[Q, Q^{ut}] = [Q^u, Q^t]^u = 1$ and $[Q^v, Q^{ut}] = [Q^{vu}, Q^t]^u = 1$. Therefore, $[G_1, G_1^t] = 1$ and G_0 is a central product of G_1 and G_1^t.

By (4.2), $\langle t\rangle$ is a Sylow 2-subgroup of $C(L) \cap N(G_0)$. As G_0 contains $\langle S, S^u, S^v\rangle = L$, $C(G_0)$ is a normal subgroup of $C(L) \cap N(G_0)$ and furthermore, $t \notin C(G_0)$. Hence $C(G_0)$ is of odd order.

By (7.2), N_3' is perfect and so contained in G_1'. Since $u \in G_0 \rhd G_1'$, we have $G_1' = G_1$. Set $L_0 = \{xx^t \mid x \in G_1\}$ and $Z = G_1 \cap G_1^t$. Then $C_{G_0}(t) = L_0 C_Z(t)$. Moreover, the map defined by $x \mapsto xx^t$ for $x \in G_1$ is a homomorphism of G_1 onto L_0 whose kernel is a subgroup of Z. Then $G_1 = G_1'$ implies $L_0 = L_0'$. Since $L \leqslant C_{G_0}(t)$ and $H^{(\infty)} = L$, we conclude that $L = L_0$ and $G_1/Z(G_1)$ is isomorphic to $PSU(5, 2)$. As the Schur multiplier of $PSU(5, 2)$ is trivial, $Z(G_1) = 1$. This completes the proof of (7.4).

The remainder of this section is devoted to proving that G_0 is a normal subgroup of G.

(7.5) *If $t \in N(G_0)^g$ for an element g of G, then $g \in N(G_0)$.*

Proof. It follows from (4.9) that $N(R) = N_H(S)W$, so $N(R)$ normalizes P. Since $G_0 = \langle P, L\rangle$, we have $N(R) \leqslant N(G_0)$.

Suppose $t \in N(G_0)^g$. Then t acts on the set $\{G_1^g, G_1^{tg}\}$. If t normalizes G_1^g, then (3.10) shows that $m(G_1^g \cap H) \geqslant 3$. Likewise $m(G_1^{tg} \cap H) \geqslant 3$ and so $m(G_0^g \cap H) \geqslant 6$. But this is impossible since $m(H) = 5$. Thus $G_1^{gt} = G_1^{tg}$ and in particular, $G_0^g \cap H = L$ since $H^{(\infty)} = L$. Let P_0 be a $\langle t\rangle$-invariant Sylow 2-subgroup of G_0^g containing S. Then $N_{P_0}(R) > S = P_0 \cap H$, so [1, Lemma 2.5] shows that $\langle N_{P_0}(R), L\rangle = G_0^g$. Thus $N(R) \leqslant N(G_0)$ implies $G_0^g \leqslant N(G_0)$. Since $C(G_0)$ has odd order, we conclude that $G_0^g = G_0$.

(7.6) *P is a Sylow 2-subgroup of $G^{(\infty)}$.*

Proof. Let P_1 be a Sylow 2-subgroup of $N(G_0)$ containing $P\langle t\rangle$ and set $P_0 = N_{P_1}(G_1)$, $Q_0 = C_{P_1}(G_1^t)$, and $X = N(G_0)$. Then we have $| X : N_X(G_1)| = 2$ and $P_1 = P_0\langle t\rangle$. If $g \in N(P_1)$, then $t^g \in P_1 \leqslant X$ and $g \in X$ by (7.5). Thus $N(P_1) \leqslant X$ and

$$P_1 \in \mathrm{Syl}_2(G).$$

Similarly, if $t^g \in P_0$ for some $g \in G$, then $g \in X$. But this is impossible since $t \notin N_X(G_1) \lhd X$. Thus

$$t^G \cap P_0 = \varnothing.$$

If $P = P_0$, then $P \in \mathrm{Syl}_2(G')$ by the Thompson transfer lemma. Therefore, we assume $P < P_0$. As $C(G_0)$ has odd order, it follows that $Q_0 \cap Q_0^t = 1$ and Q_0 is isomorphic to a subgroup of $\mathrm{Aut}(G_1)$. So $| Q_0 : Q | \leqslant 2$. Likewise

$P_0/Q_0{}^t$ is isomorphic to a subgroup of $\mathrm{Aut}(G_1)$ and $| P_0 : QQ_0{}^t | \leqslant 2$. Thus one of the following two cases occurs:

Case 1. $| Q_0 : Q | = 2$ and $P_0 = Q_0 \times Q_0{}^t$.

Case 2. $Q_0 = Q$ and $| P_0 : P | = 2$.

Let x be an involution in P_1. If $x \notin P_0$, then x interchanges G_1 and $G_1{}^t$ and so $Q^x = Q^t$. This implies $C_P(x) \simeq Q$. Now $|(P_1/P) \cap C(x)| = 4$, so we get $| C_{P_1}(x)| \leqslant 2^{12}$. If $x \in P_0 - (Q_0P \cup Q_0{}^tP)$, then x acts both on G_1 and on $G_1{}^t$ as an outer automorphism so that $| C_P(x)| \leqslant 2^8$ and $| C_{P_1}(x)| \leqslant 2^{11}$. Therefore,

$$| C_{P_1}(x)| \leqslant 2^{12}$$

for each involution x in $P_1 - (Q_0P \cup Q_0{}^tP)$. As $P = Q \times Q^t$, every involution of Pt is conjugate to t in $P\langle t\rangle$. Hence $N_{P_0}(P\langle t\rangle) = C_{P_0}(t)P$, and so $C_{P_0}(t) > S$. Moreover, $C_{P_0}(t) \cap C(L) = 1$ since $C_H(L) = \langle t\rangle\, O(H)$. Thus $C_{P_0}(t)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}(L)$ and we can take an involution a such that

$$C_{P_0}(t) = \langle a\rangle S \neq S.$$

Suppose a is conjugate in G to an element of Q_0P and choose $a^g \in Q_0P$ so that $| C_{P_1}(a^g)|$ is maximal. Then $|\langle a^g\rangle\, G_0 \cap C(a^g)|_2 \geqslant 2^{13}$, and so $| C_{P_1}(a^g)| \geqslant 2^{13}$ by the choice of a^g. Hence a^g is extremal in P_1 with respect to G. By (2.2), there exists an element h such that $a^h = a^g$ and $C_{P_1}(a)^h \leqslant C_{P_1}(a^g)$. As t lies in $C_{P_1}(a)$, (7.5) implies that $h \in X$. In particular, h normalizes the set $I = G_1C_X(G_1) \cup G_1{}^tC_X(G_1{}^t)$. Now $[t, a] = 1$ and $a \notin G_0C_X(G_0) = G_1C_X(G_1) \cap G_1{}^tC_X(G_1{}^t)$, so that $a^h \notin I$. This, however, is a contradiction since $a^g \in G_1{}^tC_X(G_1{}^t)$. Thus

$$a^G \cap (Q_0P \cup Q_0{}^tP) = \varnothing.$$

In Case 1, P_1/P is dihedral of order 8, $\mathscr{E}^*(P_1/P) = \{P_0/P, \langle t, a\rangle P/P\}$, and $Z(P_1 \bmod P) = \langle a\rangle P$. Moreover, as shown above $t^G \cap P_0 = \varnothing$ and $a^G \cap P_0 \leqslant aP$. Thus $P \in \mathrm{Syl}_2(G'')$ by [8, Lemma (1G)].

Now assume Case 2. If $(ta)^G \cap P \neq \varnothing$, choose $(ta)^g \in P$ so that $| C_{P_1}((ta)^g)|$ is maximal. Then as before $(ta)^g$ is extremal in P_1 with respect to G and we can take an element h such that $(ta)^h = (ta)^g$ and $C_{P_1}(ta)^h \leqslant C_{P_1}((ta)^g)$. Then as $t \in C_{P_1}(ta)$, (7.5) forces $h \in X$. However, as $ta \in P_1 - P_0$, this implies $(ta)^h \notin G_0$, a contradiction. Therefore, $(ta)^G \cap P = \varnothing$. Now $P_0 = \langle a\rangle P$ and $P_1 = \langle t, a\rangle P$. Thus $P \in \mathrm{Syl}_2(G')$ by [8, Lemma (1F)]. The proof is complete.

(7.7) $E(G) \simeq PSU(5, 2) \times PSU(5, 2)$.

Proof. As Y is elementary abelian of order 2^8, (3.7)(2) implies $Y = J_e(P) = J_e(Q) \times J_e(Q)^t$. If B is a subgroup of P such that $| B | \geqslant 2^{16}$ and $| \Omega_1(Z(B))| \geqslant 2^8$, then $Z(B) \geqslant Y$ and $B \leqslant C_P(Y) = D_2$ by (4.21), so $B = D_2$. Hence, D_2 is weakly closed in P with respect to G.

Let $X = G^{(\infty)}$ and suppose $d^X \cap (P - Q) \neq \varnothing$ for some involution $d \in Q$. Since every involution of G_1 is conjugate in G_1 to an element of $J_e(Q)$, we may assume that $d \in J_e(Q)$ and $d^g \in Y - J_e(Q)$ for some $g \in X$. Now $N_X(D_2) \trianglerighteq N_2 = N_3' \times N_3'^t$, so N_3' is normal in $N_X(D_2)$ by the Krull–Schmidt theorem. Furthermore, $N_X(D_2)$ controls the fusion of elements of Y by the above. But then $d^g \in Y \cap N_3' = J_e(Q)$, a contradiction. Thus Q is strongly involution closed in P with respect to X. Setting $N = \langle I(Q)^X \rangle$, we have $[N, N^t] \leqslant O(X)$ by [15]. Moreover, $N = \langle G_1{}^X \rangle$ and $N = N'$. Let $\bar{G} = G/O(G)$. Then $[\bar{N}, \bar{N}^t] = 1$ and $C_{\bar{X}}(\bar{t})^{(\infty)} = \bar{L}$. Now, it follows that $\bar{N} \cong PSU(5, 2)$ as in the last part of the proof of (7.4). Hence $\bar{N} = \bar{G}_1$ and $\bar{G}_0 \lhd \bar{X}$. Therefore, [20, Lemma (2.10)] shows that $E(G) = G_0$. The proof is complete.

8. Proof of the Theorem

Let $\bar{G} = G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ and $C_{\bar{G}}(\bar{L})$ has cyclic Sylow 2-subgroups. Suppose $\bar{L} \ntrianglelefteq \bar{G}$. Then $\bar{G}$ satisfies Hypothesis (4.1) and $E(\bar{G})$ is isomorphic to $PSL(5, 4)$ or $PSU(5, 2) \times PSU(5, 2)$ by (5.2), (6.22), and (7.7). Therefore, $E(G)$ is isomorphic to $PSL(5, 4)$ or $PSU(5, 2) \times PSU(5, 2)$ by [20, Lemma (2.10)]. The proof of the theorem is complete.

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